Families of Divisors on T-Varieties and Exceptional Sequences on $\mathbb{C}^*$-Surfaces

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Abstract

We show how one-parameter homogeneous deformations of rational T-varieties induce maps from a subgroup of the Picard group of any fiber of the deformation to the Picard group of the special fiber. If the special fiber is complete, this map preserves Euler characteristic and intersection numbers, and if the deformation is locally trivial, then this map is an isomorphism. We offer a simple description of this map for smooth, complete rational $\mathbb{C}^*$-surfaces. These results are then applied to analyze the behaviour of exceptional sequences of lines bundles on rational $\mathbb{C}^*$-surfaces under deformation and degeneration. We also show that all rational $\mathbb{C}^*$-surfaces of fixed Picard number can be connected by homogeneous deformations.

Keywords: Toric varieties, deformation theory, T-varieties, exceptional sequences

MSC: Primary 14D15, 14M25, 18E30.

Introduction

Normal rational varieties admitting an effective codimension-one torus action provide a natural generalization of toric varieties. Such varieties, also called rational complexity-one T-varieties, can be described combinatorially in terms of polyhedral divisors, see [AHS08]. Recently, R. Vollmert and the second author showed how to construct homogeneous deformations of these varieties using Minkowski decompositions in [IV09]. The goal of this paper is to use the homogeneous structure of these deformations to compare Cartier divisors on the fibers. After handling the general case, we pay special attention to rational $\mathbb{C}^*$-surfaces, and show how our results can be applied to better understand exceptional sequences.

Consider now a homogeneous one-parameter deformation $\pi : X^{\mathrm{tot}} \to S$ of a rational complexity-one T-variety X_0. For each $s \in S$ we will construct a natural subgroup $\mathrm{Pic}'(X_s) \subset \mathrm{Pic}(X_s)$. In particular, if $s = 0$ or π is locally trivial, we will have that $\mathrm{Pic}'(X_s) = \mathrm{Pic}(X_s)$. Our first main result can then be summed up in the following theorem:

Main Theorem 1. *For any $s, s' \in S$ with $s \neq 0$, π induces a natural injection $\bar{\pi}_{s,s'}$: $\mathrm{Pic}'(X_s) \hookrightarrow \mathrm{Pic}'(X_{s'})$. Furthermore, $\bar{\pi}_{s,s'}$ preserves Euler characteristic and intersection numbers if X_0 is complete. Finally, if X_0 is complete and π locally trivial, then $\bar{\pi}_{s,s'}$ is in fact an isomorphism.*

We then turn our attention to homogeneous deformations of smooth, complete, rational $\mathbb{C}^*$-surfaces. Such deformations can be blown up and down in a natural manner, as we shall see. Furthermore, there is an easy description of many such deformations where all fibers are toric surfaces. An important result is that all rational $\mathbb{C}^*$-surfaces of fixed Picard number can be connected by a series of homogeneous deformations and degenerations. In the setting of homogeneous deformations of $\mathbb{C}^*$-surfaces, we then give an explicit description of the above map $\bar{\pi}_{s,0}$ of divisors and show that this commutes with blowing up.

Finally, we look at exceptional sequences of line bundles on rational surfaces, in particular, those with $\mathbb{C}^*$ action. Important for the study of such exceptional sequences are so-called *toric systems*, defined by L. Hille and M. Perling in [HP08]. A toric system consists of a tuple of line bundles satisfying certain conditions regarding the intersection numbers, see section 4.1 for an exact definition; in any case, any full exceptional sequence of line bundles gives rise to a toric system. In [HP08] it was shown how for any toric system $\mathcal{A}$ on a rational surface X, one can construct an associated smooth toric variety $\mathrm{TV}(\mathcal{A})$. A. Bondal loosely conjectured that the step from X to $\mathrm{TV}(\mathcal{A})$ has something to do with degeneration. The following theorem offers a concrete result in this direction:

Main Theorem 2. *Let X and X' be two smooth, complete rational $\mathbb{C}^*$-surfaces both with Picard number $\rho > 2$ and let $\mathcal{A}$ be toric system on X. Then there is a sequence*

$$X = X^0 \dashrightarrow X^1 \dashrightarrow \cdots \dashrightarrow X^k = X'$$

of homogeneous deformations and degenerations connecting X and X' such that if $\mathcal{A}_i$ is the image of $\mathcal{A}$ on X^i, $\mathcal{A}_i$ is a toric system. Furthermore $\mathrm{TV}(\mathcal{A}_i) = \mathrm{TV}(\mathcal{A})$ *for all i.*

The downside to the above theorem is that not every toric system comes from an exceptional sequence. In fact, exceptional sequences do not in general degenerate to exceptional sequences. However, for a certain subset of exceptional sequences, we can describe what happens: using the inductive process of augmentation from [HP08], we define *tame* toric systems. Every tame toric system comes from an exceptional sequence, and Hille and Perling conjectured that these are in fact all possible toric systems coming from exceptional sequences. In passing, we show that for any rational surface X of fixed Picard number and any toric surface Y with equal Picard number, there exists a tame toric system $\mathcal{A}$ on X with $\mathrm{TV}(\mathcal{A}) = Y$. Given a tame toric system on some $\mathbb{C}^*$-surface along with a degeneration, we then formulate a condition of *compatibility*, which can be checked recursively. The following theorem makes clear the importance of this condition:

Main Theorem 3. *Let π be a homogeneous deformation of rational $\mathbb{C}^*$-surfaces with general fiber X_s and let $\mathcal{A}$ be a toric system on X_s. Then $\mathcal{A}$ is compatible with π if and only if $\bar{\pi}_{s,0}(\mathcal{A})$ is a tame toric system.*

In the case of Hirzebruch surfaces, we can also describe what happens under deformation or degeneration in terms of so-called *mutations*. We also use the homogeneous geometric deformations to construct noncommutative deformations, that is, parametrizations of derived categories of rational surfaces; several such parametrizations have already been described in [Per09].

The necessary language of T-varieties and polyhedral divisors is presented in section 1. Section 2.1 then recalls the notion of a homogeneous deformation. In section 2.2, we develop the machinery necessary to describe the families of divisors corresponding to a homogeneous deformation, whereas in section 2.3 we prove the surjectivity of the corresponding map of Picard groups. In sections 3.1, 3.2, 3.3, and 3.4 we concentrate on rational $\mathbb{C}^*$-surfaces, respectively introducing multidivisors to describe these surfaces, analyzing deformations of them via degeneration diagrams, proving that they are homogeneously deformation connected, and explicitly describing the map of Picard groups. We then turn to basics of exceptional sequences and toric systems in section 4.1, considering augmentations and tame toric systems in section 4.2. Finally, in section 4.3 we consider the connection between homogeneous deformations and toric systems, and in section 4.4 we use this to construct noncommutative deformations.

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1 Polyhedral Divisors and T-Varieties

We recall here the basic construction of T-varieties as well as several other facts. Unless otherwise noted, all statements can be found in [AHS08]. For a more detailed exposition and numerous examples, the reader may refer to this source.

As usual, let N be a lattice with dual M and let $N_\mathbb{Q}$ and $M_\mathbb{Q}$ be the associated $\mathbb{Q}$ vector spaces. For any polyhedron $\Delta \subset N_\mathbb{Q}$, let $\mathrm{tail}(\Delta)$ denote its tailcone, that is, the cone of unbounded directions in Δ. Thus, Δ can be written as the Minkowski sum of some bounded polyhedron and its tailcone. Now for $u \in \mathrm{tail}(\Delta)^\vee \cap M$, denote by $\mathrm{face}(\Delta, u)$ the face of Δ upon which u achieves its minimum. For any polyhedral complex C in $N_\mathbb{Q}$, let $|C|$ denote all points of $N_\mathbb{Q}$ contained in some element of C.

Now let Y be a smooth projective variety over $\mathbb{C}$ and let $\delta \subset N_\mathbb{Q}$ be a pointed polyhedral cone. A *polyhedral divisor* on Y with tail cone δ is then defined to be a formal finite sum

$$\mathcal{D} = \sum_P \Delta_P \cdot P,$$

where P runs over all prime divisors on Y and Δ_P is a polyhedron with tailcone δ. Here, finite means that only finitely many coefficients differ from the tail cone. Note that the empty set is also allowed as a coefficient.

We can evaluate such a polyhedral divisor for every element $u \in \delta^\vee \cap M$ via

$$\mathcal{D}(u) := \sum_P \min_{v \in \Delta_P} \langle v, u \rangle P$$

in order to obtain an ordinary $\mathbb{Q}$-divisor on $\mathrm{Loc}\,\mathcal{D}$, where $\mathrm{Loc}\,\mathcal{D} := Y \setminus \left(\bigcup_{\Delta_P = \emptyset} P\right)$. A polyhedral divisor $\mathcal{D}$ is called *proper* if for all $u \in \delta^\vee \cap M$, $\mathcal{D}(u)$ is semiample (i.e. a multiple is globally generated) and if for all $u \in \mathrm{relint}\,\delta^\vee \cap M$, $\mathcal{D}(u)$ is big (i.e. a multiple has a section with affine complement). To a proper polyhedral divisor we associate an M-graded $\mathbb{C}$-algebra and consequently an affine scheme admitting a $T^N = N \otimes \mathbb{C}^*$-action:

$$X(\mathcal{D}) := \mathrm{Spec} \bigoplus_{u \in \delta^\vee \cap M} H^0(Y, \mathcal{D}(u)).$$

This construction gives a normal variety of dimension $\dim Y + \dim N_{\mathbb{Q}}$ together with a T^N-action.

We now wish to glue these affine schemes together. For any two proper polyhedral divisors $\mathcal{D} = \sum_P \Delta_P \cdot P$, $\mathcal{D}' = \sum_P \Delta'_P \cdot P$ on Y with tail cones δ and δ' we define their *intersection* by

$$\mathcal{D} \cap \mathcal{D}' := \sum_P (\Delta_P \cap \Delta'_P) \cdot P.$$

Likewise, we say $\mathcal{D}' \subset \mathcal{D}$ if $\Delta'_P \subset \Delta_P$ for every prime divisor $P \in Y$. If $\mathcal{D}' \subset \mathcal{D}$ then we have an inclusion

$$\bigoplus_{u \in \delta^\vee \cap M} H^0(Y, \mathcal{D}'(u)) \supset \bigoplus_{u \in \delta^\vee \cap M} H^0(Y, \mathcal{D}(u))$$

which corresponds to a dominant morphism $X(\mathcal{D}') \to X(\mathcal{D})$. If this is an open embedding, we say $\mathcal{D}'$ is a *face* of $\mathcal{D}$ and write $\mathcal{D}' \prec \mathcal{D}$.[1] We now define a *divisorial fan* to be a finite set Ξ of proper polyhedral divisors such that for $\mathcal{D}, \mathcal{D}' \in \Xi$ we have $\mathcal{D} \succ \mathcal{D}' \cap \mathcal{D} \prec \mathcal{D}'$ with $\mathcal{D}' \cap \mathcal{D}$ also in Ξ. We may then glue the affine varieties $X(\mathcal{D})$ for $\mathcal{D} \in \Xi$ via

$$X(\mathcal{D}) \leftarrow X(\mathcal{D} \cap \mathcal{D}') \rightarrow X(\mathcal{D}').$$

This construction yields a normal scheme $X(\Xi)$ of dimension $\dim Y + \dim N_{\mathbb{Q}}$ with a torus action by T^N. Note that all normal varieties with torus action can be constructed in this manner.

For a not-necessarily closed point $y \in Y$ and a polyhedral divisor $\mathcal{D} = \sum \Delta_P \cdot P$, set $\mathcal{D}_y := \sum_{y \in P} \Delta_P$. Likewise, for a divisorial fan Ξ, the polyhedral complex defined by the $\mathcal{D}_y$, $\mathcal{D} \in \Xi$ is called a *slice* of Ξ. Note that $X(\Xi)$ is complete if and only if all slices Ξ_y are complete subdivisions of $N_{\mathbb{Q}}$. We further define $\mathrm{tail}(\Sigma) = \{\mathrm{tail}(\mathcal{D}) \mid \mathcal{D} \in \Xi\}$; this set of cones forms a fan which we call the *tailfan* of Ξ.

We will be dealing extensively with invariant divisors on complete T-varieties with codimension one torus action; these divisors have been described in [PS08] in combinatorial terms. Let Y be a smooth projective curve and Ξ a divisorial fan on Y; set $\Sigma = \mathrm{tail}(\Xi)$. We then define $\mathrm{SF}(\Xi)$ be the set of all formal sums of the form

$$h = \sum_{P \in Y} h_P \otimes P$$

where $h_P : |\Xi_P| \to \mathbb{Q}$ are continuous functions such that:

(i) h_P is piecewise affine with respect to the subdivision Ξ_P;

(ii) h_P is integral, that is, if $k \cdot v$ is a lattice point for $k \in \mathbb{N}$, $v \in N$, then $k \cdot h_P(v) \in \mathbb{Z}$;

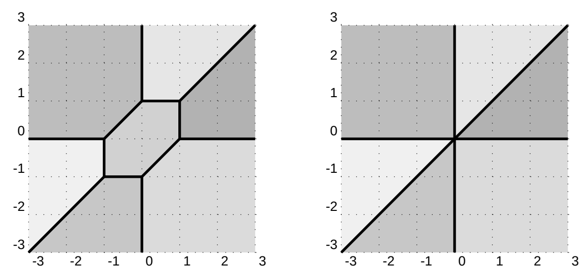
(iii) For $v \in |\Sigma|$, $h_P^0 := \lim_{k \to \infty} h_P(k \cdot v + v_P)/k$ is the same for all P, where v_P is any point in $\mathcal{D}_P$ for some $\mathcal{D} \in \Xi$ with $v \in \mathrm{tail}\,\mathcal{D}$. We call h_P^0 the linear part of h and denote it by h^0;

(iv) $h_P \neq h^0$ for only finitely many P.

[1] This condition can be made more explicit, see definition 5.1 of [AHS08].

Furthermore, let $\mathrm{CaSF}(\Xi)$ consist of all $h \in \mathrm{SF}(\Xi)$ such that for every $\mathcal{D} \in \Xi$ with complete locus, any linear extension of $h_{|\mathcal{D}}$ evaluated at 0 is principal on Y. Both $\mathrm{SF}(\Xi)$ and $\mathrm{CaSF}(\Xi)$ have a natural group structure. There is a group isomorphism from $\mathrm{CaSF}(\Xi)$ to the group $\text{T-CaDiv}(X(\Xi))$ of T-invariant Cartier divisors on $X(\Xi)$; we denote the divisor associated to h by D_h. Under this isomorphism, linear functions h with $h(0)$ a principal divisor on Y are taken to principal divisors.

Example. The following example is considered in [Süß08], example 5.1. Consider $X = \overline{\mathrm{Cone}(dP_6)}$, a compactification of the cone over the del Pezzo surface of degree six. X is in fact toric, and by restricting to a subtorus we can view X as a complexity-one T-variety with divisorial fan Ξ on $Y = \mathbb{P}^1$ as pictured in figure 1. Ξ has nontrivial slices in 0 and ∞.



(a) Ξ_0 (b) Ξ_∞

Figure 1: Divisorial fan slices for $\overline{\mathrm{Cone}(dP_6)}$

The noncompact polyhedra with common tail cone belong to the same polyhedral divisor, whereas the polyhedral divisor with hexagonal coefficient at 0 has the empty set as coefficient at ∞.

2 Families of Divisors

2.1 Deformations of T-Varieties

Here we summarize the construction of deformations found in [IV09].[2] We shall make several simplifying assumptions.

Let $Y = \mathbb{P}^1$ and let Ξ be a divisorial fan on Y. We now show how to construct certain one-parameter homogeneous deformations of the rational non-affine T-variety $X(\Xi)$. Fix some point $0 \in Y$ such that the slice Ξ_0 is non-trivial, and let $\mathcal{P} \subset Y$ be the set of all points P with non-trivial slice Ξ_P. In particular, we have $0 \in \mathcal{P}$.

Definition. A 1-*parameter Minkowski decomposition* of the slice Ξ_0 consists of polyhedra $\mathcal{D}_0^0, \mathcal{D}_0^1$ for every $\mathcal{D} \in \Xi$ such that:

(i) For $\mathcal{D} \in \Xi$, the slice $\mathcal{D}_0 = \mathcal{D}_0^0 + \mathcal{D}_0^1$;

(ii) $\mathcal{D}_0^0, \mathcal{D}_0^1$ have the same tailcone as $\mathcal{D}$;

[2] At least for toric varieties, an alternate construction of deformations by using homogeneous coordinates can be found in [Mav09].

(iii) For fixed $j = 0, 1$, $\{\mathcal{D}_0^j\}_{\mathcal{D}\in\Xi}$ form a polyhedral complex in $N_\mathbb{Q}$;

(iv) If $\mathcal{D} = \mathcal{D}' \cap \mathcal{D}''$ for $\mathcal{D}', D'' \in \Xi$, then $\mathcal{D}_0^j = \mathcal{D}_0'^j \cap \mathcal{D}_0''^j$ for $j = 0, 1$.

For fixed $j = 0, 1$, each vertex v of the polyhedral complex Ξ_0 corresponds to exactly one vertex v_j of the polyhedral complex $\{\mathcal{D}_0^j\}$. We say that the decomposition of Ξ_0 is *admissible* if and only if for all vertices v of Ξ_0, at most one of the corresponding vertices v_j is not a lattice point.

Let $\{\mathcal{D}_0^j\}_{\mathcal{D}\in\Xi}$ be an admissible decomposition of Ξ_0. Set $S^* = \mathbb{A}^1 \setminus \mathcal{P}$ and let $S = S^* \cup \{0\}$. From such data we shall construct a divisorial fan Ξ^{tot} on $Y^{\mathrm{tot}} := Y \times S$. For $P \in \mathcal{P}$, let $D^{\mathrm{tot}}(P)$ be the prime divisor $P \times S$ on $Y^{\mathrm{tot}} = Y \times S$ and let $D^{\mathrm{tot}}(\Delta)$ be the divisor given by the vanishing of $y - t$, where y is a local parameter for 0 in Y and t the coordinate on $\mathbb{A}^1$. For $\mathcal{D} \in \Xi$ we can define a polyhedral divisor $\mathcal{D}^{\mathrm{tot}}$ on Y^{tot} by

$$\mathcal{D}^{\mathrm{tot}} = \mathcal{D}_0^0 \cdot D^{\mathrm{tot}}(0) + \mathcal{D}_0^1 \cdot D^{\mathrm{tot}}(\Delta) + \sum_{P\in\mathcal{P}\setminus\{0\}} \mathcal{D}_P \cdot D^{\mathrm{tot}}(P).$$

We then set $\Xi^{\mathrm{tot}} = \langle \{\mathcal{D}^{\mathrm{tot}}\}_{\mathcal{D}\in\Xi} \rangle$, that is, Ξ^{tot} is the set of polyhedral divisors induced by the $\mathcal{D}^{\mathrm{tot}}$ via intersection.

We also construct a family of divisorial fans $\Xi^{(s)}$ on Y. Fix a point $s \in S$. For $\mathcal{D} \in \Xi$ we can define a polyhedral divisor $\mathcal{D}^{(s)}$ on Y by

$$\mathcal{D}^{(s)} = \mathcal{D}_0^0 \cdot 0 + \mathcal{D}_0^1 \cdot s + \sum_{P\in\mathcal{P}\setminus\{0\}} \mathcal{D}_P \cdot P$$

where the coefficients in front of prime divisors appearing multiple times are added via Minkowski sums. In particular, $\mathcal{D}^{(0)} = \mathcal{D}$. Similar to above, we then set $\Xi^{(s)} = \langle \{\mathcal{D}^{(s)}\}_{\mathcal{D}\in\Xi} \rangle$. Note that both Ξ^{tot} and $\Xi^{(s)}$ are divisorial fans. Furthermore, Ξ^{tot} comes with a rational quotient map $\Xi^{\mathrm{tot}} \dashrightarrow Y^{\mathrm{tot}}$, which when composed with the projection onto S can in fact be (uniquely) extended to a regular map $\pi\colon X(\Xi^{\mathrm{tot}}) \to S$. This gives a one-parameter deformation of $X(\Xi)$:

Theorem 2.1. *The map $\pi\colon X(\Xi^{\mathrm{tot}}) \to S$ gives a flat family with $\pi^{-1}(s) \cong X(\Xi^{(s)})$ for $s \in S$. In particular, $\pi^{-1}(0) = X(\Xi)$.*

We will refer to π as a one-parameter homogeneous deformation of $X(\Xi)$. The following proposition will be used in the next section:

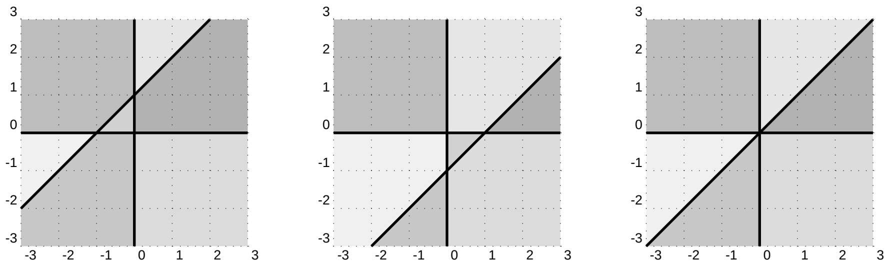
Proposition 2.2. *Suppose that $X(\Xi)$ is a complete variety and let π be a one-parameter homogeneous deformation of $X(\Xi)$. Then π is proper.*

Proof. First, note that $X(\Xi)$ being complete implies that the slices Ξ_y are complete subdivisions of $N_\mathbb{Q}$ for all $y \in Y$. It easily follows that Ξ_y^{tot} are complete subdivisions of $N_\mathbb{Q}$ for all $y \in Y \times S$.

The map π is in fact a torus equivariant morphism corresponding to the triple $(\mathrm{pr}, F, 0)$, where $\mathrm{pr} : Y \times S \to S$ is the projection and $F : N \to 0$ is the zero map, see [Süß09]. Since pr is proper and all slices of Ξ^{tot} cover $N_\mathbb{Q}$, the properness of π follows from theorem 7.1 of [Süß09]. □

Remark. It was shown in [IV09] that if $X_0 = X(\Xi)$ is smooth, complete, and toric, the set of all one-parameter homogeneous deformations of X_0 span the space of infinitesimal deformations of X_0.

Example. We return to the example from section 1. As shown in [Süß08], there is a homogeneous deformation of $X = \overline{\mathrm{Cone}(dP_6)}$ to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The three nontrivial slices of the divisorial fan Ξ^{tot} for the total space are pictured in figure 2. Ξ^{tot} has nontrivial slices for



(a) $\Xi^{\mathrm{tot}}_{D^{\mathrm{tot}}(0)}$ (b) $\Xi^{\mathrm{tot}}_{D^{\mathrm{tot}}(\Delta)}$ (c) $\Xi^{\mathrm{tot}}_{D^{\mathrm{tot}}(\infty)}$

Figure 2: A deformation of $\overline{\mathrm{Cone}(dP_6)}$

the divisors $D^{\mathrm{tot}}(0)$, $D^{\mathrm{tot}}(\Delta)$, and $D^{\mathrm{tot}}(\infty)$. In these slices, the noncompact polyhedra with common tail cone belong to the same polyhedral divisor, whereas the triangular coefficients at $D^{\mathrm{tot}}(0)$, $D^{\mathrm{tot}}(\Delta)$ belong to a common polyhedral divisor having the empty set as coefficient at $D^{\mathrm{tot}}(\infty)$. One easily checks that the general fiber of this deformation is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

2.2 Induced Families of Divisors

Let $\pi : X(\Xi^{\mathrm{tot}}) \to S$ be a one-parameter homogeneous deformation of $X(\Xi)$. Denote by $X_s = X(\Xi^{(s)})$ the fiber over $s \in S$, and set $X^{\mathrm{tot}} = X(\Xi^{\mathrm{tot}})$. The basic observation is that certain T-invariant divisors on the total space X^{tot} restrict to T-invariant divisors on each fiber X_s, thus giving us a family of T-invariant divisors. We wish to analyze the situation more closely in this section.

For $s \in S \subset \mathbb{P}^1$, let $\mathrm{CaSF}'(\Xi^{(s)})$ consist of those $h \in \mathrm{CaSF}(\Xi^{(s)})$ such that for all $\mathcal{D}^{(s)} \in \Xi^{(s)}$, we can find $u \in M$ and $a_0, a_s \in \mathbb{Z}$ such that $(h_{|\mathcal{D}^{(s)}})_0(v) = \langle v, u\rangle + a_0$ and $(h_{|\mathcal{D}^{(s)}})_s(v) = \langle v, u\rangle + a_s$. Note that $\mathrm{CaSF}'(\Xi^{(s)}) = \mathrm{CaSF}(\Xi^{(s)})$ if π is locally trivial, which in particular is the case if X_0 is smooth. We of course also always have $\mathrm{CaSF}'(\Xi^{(0)}) = \mathrm{CaSF}(\Xi^{(0)})$.

Fix now some $s \in S^*$ and choose some support function $h^{(s)} \in \mathrm{CaSF}'(\Xi^{(s)})$; this corresponds to an invariant Cartier divisor $D_{h^{(s)}}$ on X_s. For each $s' \in S^*$, define $h^{(s')} \in \mathrm{CaSF}(\Xi^{(s')})$ by $h^{(s')} = h^{(s)} + h_s^{(s)} \otimes (s' - s)$. Likewise, define $h^{(0)} \in \mathrm{CaSF}(\Xi^{(0)})$ by setting $h_P^{(0)} = h_P^{(s)}$ for $P \in Y \setminus \{0, s\}$, taking $h_s^{(0)}$ to be trivial, and for $\mathcal{D} \in \Xi$ and any $v \in \mathcal{D}_0$, setting

$$h_0^{(0)}(v) = h_0^{(s)}(v_0) + h_s^{(s)}(v_s)$$

where $v_0 \in \mathcal{D}_0^{(s)}$ and $v_s \in \mathcal{D}_0^{(s)}$ are such that $v_0 + v_s = v$. Note that the requirement $h^{(s)} \in \mathrm{CaSF}'(\Xi^{(s)})$ ensures that $h_0^{(0)}(v)$ does not depend on the choice of such v_0 and v_s.

We also define an invariant Cartier divisor D_h^{tot} on X^{tot}. We first will need invariant open coverings of X_s and X^{tot}. For $P \in \mathcal{P} \setminus \{0\}$ and $\mathcal{D} \in \Xi$ with noncomplete locus, set

$$U_{\mathcal{D},P}^{(s)} = X\Big(\mathcal{D}^{(s)} + \emptyset \cdot s + \sum_{\substack{Q \in \mathcal{P} \\ Q \neq P}} \emptyset \cdot Q\Big)$$

$$U_{\mathcal{D},P}^{\mathrm{tot}} = X\Big(\mathcal{D}^{\mathrm{tot}} + \emptyset \cdot D(\Delta) + \sum_{\substack{Q \in \mathcal{P} \\ Q \neq P}} \emptyset \cdot D^{\mathrm{tot}}(Q)\Big)$$

and likewise set

$$U_{\mathcal{D},0}^{(s)} = X\Big(\mathcal{D}^{(s)} + \sum_{\substack{Q \in \mathcal{P} \\ Q \neq 0}} \emptyset \cdot Q\Big)$$

$$U_{\mathcal{D},0}^{\mathrm{tot}} = X\Big(\mathcal{D}^{\mathrm{tot}} + \sum_{\substack{Q \in \mathcal{P} \\ Q \neq 0}} \emptyset \cdot D^{\mathrm{tot}}(Q)\Big).$$

On the other hand, for $P \in \mathcal{P}$ and $\mathcal{D} \in \Xi$ with complete locus, set $U_{\mathcal{D},P}^{(s)} = X(\mathcal{D}^{(s)})$ and $U_{\mathcal{D},P}^{\mathrm{tot}} = X(\mathcal{D}^{\mathrm{tot}})$. One easily checks that $\{U_{\mathcal{D},P}^{(s)}\}$ and $\{U_{\mathcal{D},P}^{\mathrm{tot}}\}$ define invariant open coverings of respectively X_s and X^{tot}. These open coverings may in fact be finer than necessary for defining the desired Cartier divisor.

For each $P \in \mathcal{P}$ and $\mathcal{D} \in \Xi$, let $u(\mathcal{D},P) \in M$, $f_{\mathcal{D},P}^{(s)} \in K(Y)$ be such that $D_{h^{(s)}}{}_{|U_{\mathcal{D},P}^{(s)}} = \operatorname{div}(f_{\mathcal{D},P}^{(s)} \cdot \chi^{u(\mathcal{D},P)})$. Such $f_{\mathcal{D},P}^{(s)}, u(\mathcal{D},P)$ exist since $h^{(s)} \in \operatorname{CaSF}'(\Xi^{(s)})$. Now set

$$f_{\mathcal{D},P}^{\mathrm{tot}} = f_{\mathcal{D},P}^{(s)} \cdot \left(\frac{y-t}{y-s}\right)^{\nu_s(f_{\mathcal{D},P}^{(s)})} \in K(Y^{\mathrm{tot}}),$$

where ν_s is the valuation in the point s and y and t are as in the previous section. This leads to the following proposition:

Proposition 2.3. *With respect to the open covering $X^{\mathrm{tot}} = \bigcup U_{\mathcal{D},P}^{\mathrm{tot}}$, the functions $f_{\mathcal{D},P}^{\mathrm{tot}} \cdot \chi^{u(\mathcal{D},P)} \in K(X^{\mathrm{tot}})$ define an invariant Cartier divisor on X^{tot} which we denote by D_h^{tot}. Furthermore, the restriction of this divisor to the fiber $X_{s'}$ for any $s' \in S$ is equal to*

$$(D_h^{\mathrm{tot}})_{s'} = D_{h^{(s')}}.$$

Proof. A simple calculation shows that the restriction of the functions $f_{\mathcal{D},P}^{\mathrm{tot}} \cdot \chi^{u(\mathcal{D},P)}$ to any fiber $X_{s'}$ give the Cartier divisor corresponding to $h^{(s')}$. We now show that these functions do indeed define a Cartier divisor on X^{tot}.

Consider $\mathcal{D}, \mathcal{D}' \in \Xi$ and $P, P' \in \mathcal{P}$. It is sufficient to show

$$\frac{f_{\mathcal{D},P}^{\mathrm{tot}}}{f_{\mathcal{D}',P'}^{\mathrm{tot}}} \cdot \chi^{u(\mathcal{D},P)-u(\mathcal{D}',P')} \in H^0\Big(U_{\mathcal{D},P}^{\mathrm{tot}} \cap U_{\mathcal{D}',P'}^{\mathrm{tot}}, \mathcal{O}_{X^{\mathrm{tot}}}\Big).$$

Setting $\tilde{u} = u(\mathcal{D}, P) - u(\mathcal{D}', P')$, this is equivalent to showing

$$g := \frac{f^{(s)}_{\mathcal{D},P}}{f^{(s)}_{\mathcal{D}',P'}} \cdot \left(\frac{y-t}{y-s}\right)^{\nu_s(f^{(s)}_{\mathcal{D},P}/f^{(s)}_{\mathcal{D}',P'})} \in H^0\Big(Y^{\mathrm{tot}}_{\mathcal{D},P} \cap Y^{\mathrm{tot}}_{\mathcal{D}',P'}, \mathcal{D}^{\mathrm{tot}} \cap \mathcal{D}'^{\mathrm{tot}}\big(\tilde{u}\big)\Big),$$

where $Y^{\mathrm{tot}}_{D,P} = Y^{\mathrm{tot}}$ if $\operatorname{Loc}\mathcal{D}$ is complete and $Y^{\mathrm{tot}}_{D,P}$ is the image of $U^{\mathrm{tot}}_{D,P}$ under the quotient map otherwise. This in turn is the same as showing that

$$\nu_D(g) \geq -(\mathcal{D}^{\mathrm{tot}} \cap \mathcal{D}'^{\mathrm{tot}})_D\big(\tilde{u}\big) \tag{1}$$

for all divisors D contained in $Y^{\mathrm{tot}}_{\mathcal{D},P} \cap Y^{\mathrm{tot}}_{\mathcal{D}',P'}$, where ν_D is the corresponding valuation. One immediately sees that this is automatically fulfilled unless D is of the form $D^{\mathrm{tot}}(Q), Q \in \mathcal{P}$, $D^{\mathrm{tot}}(s)$ or $D^{\mathrm{tot}}(\Delta)$, since both sides of the above inequality will be 0.

Now for $Q \in \mathcal{P}$, $\nu_{D^{\mathrm{tot}}(Q)}(g) = \nu_Q(f^{(s)}_{\mathcal{D},P}/f^{(s)}_{\mathcal{D}',P'})$. Furthermore, $\nu_{D^{\mathrm{tot}}(s)}(g) = 0$ and $\nu_{D^{\mathrm{tot}}(\Delta)}(g) = \nu_s(f^{(s)}_{\mathcal{D},P}/f^{(s)}_{\mathcal{D}',P'})$. On the other hand, we have

$$\begin{aligned}
(\mathcal{D}^{\mathrm{tot}} \cap \mathcal{D}'^{\mathrm{tot}})_{D^{\mathrm{tot}}(Q)}\big(\tilde{u}\big) &= (\mathcal{D}^{(s)} \cap \mathcal{D}'^{(s)})_Q\big(\tilde{u}\big);\\
(\mathcal{D}^{\mathrm{tot}} \cap \mathcal{D}'^{\mathrm{tot}})_{D^{\mathrm{tot}}(s)}\big(\tilde{u}\big) &= 0;\\
(\mathcal{D}^{\mathrm{tot}} \cap \mathcal{D}'^{\mathrm{tot}})_{D^{\mathrm{tot}}(\Delta)}\big(\tilde{u}\big) &= (\mathcal{D}^{(s)} \cap \mathcal{D}'^{(s)})_s\big(\tilde{u}\big).
\end{aligned}$$

Now, since the functions $f^{(s)}_{\mathcal{D},P}\chi^{u(\mathcal{D},P)}$ define a Cartier divisor on X_s, we have

$$\frac{f^{(s)}_{\mathcal{D},P}}{f^{(s)}_{\mathcal{D}',P'}} \in H^0\Big(Y_{\mathcal{D},P} \cap Y_{\mathcal{D}',P'}, \mathcal{D}^{(s)} \cap \mathcal{D}'^{(s)}\big(\tilde{u}\big)\Big)$$

where $Y_{D,P}$ is defined similarly to $Y^{\mathrm{tot}}_{D,P}$. Consequently,

$$\nu_Q(g) \geq -(\mathcal{D}^{\mathrm{tot}} \cap \mathcal{D}'^{\mathrm{tot}})_Q\big(\tilde{u}\big) \tag{2}$$

for $Q \in Y_{D,P}$ and inequality (1) follows for the required divisors. $\square$

Let T-CaDiv$'$ denote the image of CaSF$'$ in the group T-CaDiv of invariant Cartier divisors. Likewise, let Pic$'$ be the image of CaSF$'$ module linear equivalence. Then for each $s' \in S$, the above construction gives us a map $\pi_{s,s'} : \text{T-CaDiv}'(X_s) \to \text{T-CaDiv}'(X_{s'})$ defined by sending $D_{h^{(s)}}$ to $D_{h^{(s')}}$. For $s' \in S^*$, it is obvious from the construction that $\pi_{s,s'}$ is an isomorphism. For $s' = 0$, the matter is slightly more delicate. It is clear from construction that $\pi_{s,0}$ is a group homomorphism sending principal divisors to principal divisors, with kernel contained in the set of principal divisors. Thus, $\pi_{s,s'}$ always descends to an injective map $\bar{\pi}_{s,s'} : \operatorname{Pic}'(X_s) \hookrightarrow \operatorname{Pic}'(X'_s)$. Furthermore, we have the following theorem:

Theorem 2.4. *Suppose that Ξ_0 is complete and π is locally trivial. Then the map $\pi_{s,0}$: $\text{T-CaDiv}'(X_s) \to \text{T-CaDiv}'(X_0)$ is surjective and thus induces an isomorphism $\bar{\pi}_{s,0} : \operatorname{Pic}'(X_s) \to \operatorname{Pic}'(X_0)$. More generally, $\pi_{s,0}$ is surjective if* $\operatorname{rank}\operatorname{Pic}'(X_s) = \operatorname{rank}\operatorname{Pic}'(X_0)$.

We postpone the proof of the theorem until section 2.3. We conjecture that $\pi_{s,0}$ is in fact surjective anytime that the support of Ξ_0 is convex, with no restrictions on π. Now if the special fiber X_0 is complete, the cohomology groups of coherent sheaves on all the fibers of π have finite rank. For invertible sheaves we then have the following theorem:

Theorem 2.5. *Let X_0 be complete, with π any one-parameter homogeneous deformation. Consider some $\mathcal{L} \in \mathrm{Pic}'(X_s)$ and any $s \in S^*$. Then we have*

(i) $h^i\left(\bar{\pi}_{s,s'}(\mathcal{L})\right) = h^i\left(\mathcal{L}\right)$ for all $i \geq 0$ and any $s' \in S^*$;

(ii) $h^i\left(\bar{\pi}_{s,0}(\mathcal{L})\right) \geq h^i\left(\mathcal{L}\right)$ for all $i \geq 0$;

(iii) $\chi\left(\bar{\pi}_{s,s'}(\mathcal{L})\right) = \chi\left(\mathcal{L}\right)$ for any $s' \in S$.

Proof. Consider $h^{(s)} \in \mathrm{CaSF}'(\Xi^{(s)}$ with $D_{h^{(s)}} \cong \mathcal{L}$. Then $\mathcal{O}(D_h^{\mathrm{tot}})$ is a line bundle on X^{tot} and thus flat over S, since π is flat. One easily checks that for $s' \in S^*$, $h^i(\mathcal{O}(D_{h^{(s')}})) = h^i(\mathcal{L})$; this can be seen for example by comparing Čech cohomology. Now since X_0 is complete, we have that π is proper by proposition 2.2. The theorem then follows from the corollary in [Mum70] section II.5, since $\mathcal{O}(D_{h^{(s')}}) = \mathcal{O}(D_h^{\mathrm{tot}})_{|X_{s'}}$ for all $s' \in S$. □

Similarly, if X_0 is complete, $\pi_{s,s'}$ preserves intersection numbers:

Theorem 2.6. *Let X_0 be complete of dimension n, with π any one-parameter homogeneous deformation. Consider invariant divisors $D^1, \ldots, D^n \in \mathrm{T\text{-}CaDiv}'(X_s)$ for some $s \in S^*$. Then for all $s' \in S$, the intersection numbers*

$$\pi_{s,s'}(D^1). \cdots .\pi_{s,s'}(D^n)$$

agree.

Proof. By proposition 2.3, we can lift the divisor D^i to a divisor $\tilde{D}^i$ on X^{tot}. Define α to be the one-cycle class attained by intersecting the divisors $\tilde{D}^1, \ldots, \tilde{D}^n$. Then $\alpha_{s'}$, the restriction of α to $X_{s'}$, is the intersection of all $\pi_{s,s'}(D^i)$. Thus, $\deg(\alpha_{s'})$ is the desired intersection number. The theorem then follows from a direct application of proposition 10.2 in [Ful98]. □

Finally, $\pi_{s,s'}$ maps canonical divisors to canonical divisors:

Theorem 2.7. *Let π any one-parameter homogeneous deformation. If for some $s \in S^*$, $K \in \mathrm{T\text{-}CaDiv}'(X_s)$ is a canonical divisor on X_s, then $\pi_{s,s'}(K)$ is a canonical divisor on $X_{s'}$ for all $s' \in S$.*

Proof. If $K \in \mathrm{T\text{-}CaDiv}'(X_s)$, we can assume (after possible modification with an invariant principal divisor) that it is of the form stated in theorem 3.19 of [PS08]. Coupled with proposition 3.16 of [PS08], we have that $K = D_{h^{(s)}}$, with $h^{(s)} \in \mathrm{CaSF}'(X_s)$ defined as follows:

(i) For $P \in Y \setminus \{0, s\}$ and v a vertex in $\Xi_P^{(s)}$, $h_P^{(s)}(v) = -1 + 1/\lambda(v)$;

(ii) For $Q \in \{0, s\}$ and v a vertex in $\Xi_Q^{(s)}$, $h_Q^{(s)}(v) = 1/\lambda(v)$;

(iii) $(h^{(s)})^0$ has slope 1 along every ray of the tailfan of $\Xi^{(s)}$;

where $\lambda(v)$ is the smallest integer such that $\lambda(v)\cdot v$ is a lattice point. Indeed, this follows immediately by taking $K_Y = -\{0\}-\{s\}$ in theorem 3.19 of [PS08]. For $s' \neq 0$, it immediately follows that $D_{h^{(s')}}$ is canonical. On the other hand, one easily checks then that $h^{(0)} \in \mathrm{CaSF}(X_0)$ is the support function defined by:

(i) For $P \in Y \setminus \{0\}$ and v a vertex in $\Xi_P^{(0)}$, $h_P^{(0)}(v) = -1 + 1/\lambda(v)$;

(ii) For v a vertex in $\Xi_0^{(0)}$, $h_0^{(0)}(v) = 1 + 1/\lambda(v)$;

(iii) $(h^{(0)})^0$ has slope 1 along every ray of the tailfan of $\Xi^{(0)}$.

Indeed, (i) and (iii) are immediate, and (ii) follows from the fact that any vertex $v \in \Xi_0^{(0)}$ is the sum of vertices of $\Xi_0^{(s)}$ and $\Xi_s^{(s)}$, one of which must be a lattice point. Taking now $K_Y = -2\cdot\{0\}$, we see again from [PS08] that $D_{h^{(0)}}$ is also canonical. $\square$

Example. We return to the example of $X_0 = \overline{\mathrm{Cone}(dP_6)}$ from the previous two sections. As previously noted, we have $X_s = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We first observe that $\text{T-CaDiv}'(X_s) \cong \mathbb{Z}^3 \times \mathrm{Div}^0(\mathbb{P}^1)$. Indeed, for $a_1, a_2, a_3 \in \mathbb{Z}$, let $h^{(s)}[a_1,a_2,a_3] \in \mathrm{CaSF}'(\Xi^{(s)})$ be the support function taking respective values $-a_1, -a_2, -a_3$ on the vertices $(0,0),(0,1),(-1,0)$ of $\Xi_0^{(s)}$, taking respective values $0, a_2-a_1, a_3-a_1$ on the vertices $(0,0),(0,-1),(1,0)$ of $\Xi_s^{(s)}$ and taking value 0 on all other vertices. Note that this completely determines $h^{(s)}[a_1,a_2,a_3]$. It is then obvious that any element of $\mathrm{CaSF}'(\Xi^{(s)})$ can be written uniquely as $h^{(s)}[a_1,a_2,a_3]+P$ for some $a_1,a_2,a_3 \in \mathbb{Z}$ and $P \in \mathrm{Div}^0(\mathbb{P}^1)$. This gives the above isomorphism.

On the other hand, we also have that $\text{T-CaDiv}(X_0) = \text{T-CaDiv}'(X_0) \cong \mathbb{Z}^3 \times \mathrm{Div}^0(\mathbb{P}^1)$. Indeed, for $a_1,a_2,a_3 \in \mathbb{Z}$, let $h^{(0)}[a_1,a_2,a_3] \in \mathrm{CaSF}'(\Xi^{(s)})$ be the support function taking respective values $-a_1,-a_2,-a_3$ at $(0,0),(0,1),(-1,0)$ of $\Xi_0^{(s)}$ and with value 0 on the vertex 0 of all other slices. As before, this completely determines $h^{(0)}[a_1,a_2,a_3]$ and as above, any element of $\mathrm{CaSF}'(\Xi^{(0)})$ can be written uniquely as $h^{(0)}[a_1,a_2,a_3]+P$. Now, if we set $h^{(s)} = h^{(s)}[a_1,a_2,a_3]+P$, then one easily checks that $h^{(0)} = h^{(0)}[a_1,a_2,a_3]+P$. Thus, in this case, the map $\pi_{s,0}$ is injective and thus an isomorphism. Factoring out by linear equivalence, we then have $\mathrm{Pic}(X_0) \cong \mathrm{Pic}'(X_s) \cong \mathbb{Z}$.

Now, $-K^{(s)} := D_{h^{(s)}[2,2,2]}$ is an anticanonical divisor for X_s. Then $-K^{(0)} := D_{h^{(0)}[2,2,2]} = \pi_{s,0}(-K^{(s)})$, and one easily checks that this is in fact an anticanonical divisor for X_0. Since both X_0 and X_s are toric Fano varieties, the higher cohomology groups of $-K^{(0)}$ and $-K^{(s)}$ vanish, so that in this case we actually have $h^i(-K^{(0)}) = h^i(-K^{(s)})$ for all $i \geq 0$, in particular for $i = 0$.

Remark. If Ξ_0 is not complete, we cannot in general expect that $\pi_{s,s'}$ is surjective, even if π is locally trivial. Consider for example X_0 to be the open subset of $\overline{\mathrm{Cone}(dP_6)}$ attained by leaving out the singular T-invariant chart; the corresponding divisorial fan is then as pictured in figure 1 with the omission of the hexagon in the middle. We then consider the deformation π of X_0 gotten by restricting our previous deformation of $\overline{\mathrm{Cone}(dP_6)}$. Note that π is now locally trivial. One easily checks that with respect to this deformation, $\text{T-CaDiv}(X_s) = \text{T-CaDiv}'(X_s) \cong \mathbb{Z}^5 \times \mathrm{Div}^0(\mathbb{P}^1)$ whereas $\text{T-CaDiv}(X_0) = \text{T-CaDiv}'(X_0) \cong \mathbb{Z}^6 \times \mathrm{Div}^0(\mathbb{P}^1)$, and that $\pi_{s,0}$ isn't surjective.

2.3 Locally Trivial Deformations

In this section, we will be considering locally trivial deformations, with the goal of proving theorem 2.4. Consider some T-variety $X_0 = X(\Xi)$. As mentioned in section 5 of [IV09], locally trivial deformations correspond to especially simple decompositions of the slice Ξ_0. Indeed, for each $\mathcal{D} \in \Xi$, we have $\mathcal{D}_0 = \mathcal{D}_0^0 + \mathcal{D}_0^1$ where either

(i) $\mathcal{D}_0^0 = \mathcal{D}_0 - v$ and $\mathcal{D}_0^1 = \operatorname{tail} \mathcal{D}_0 + v$; or

(ii) $\mathcal{D}_0^0 = \operatorname{tail} \mathcal{D}_0 + v$ and $\mathcal{D}_0^1 = \mathcal{D}_0 - v$

for some $v \in N$. This is the local picture; requiring that these decompositions of the polyhedra $\mathcal{D}_0$ fit together to a decomposition of Ξ_0 adds even further constraints. Indeed, if $\mathcal{D}_0$ and $\mathcal{D}'_0$ share some compact edge, then one easily checks that if the decomposition of $\mathcal{D}_0$ is of type i for $i = 1, 2$ and some $v \in N$, then the decomposition of $\mathcal{D}'_0$ is also of type i with the same $v \in N$. For $\mathcal{D} \in \Xi$, define the set $C(\mathcal{D}) \subset \Xi_0$ to be the unique subcomplex of Ξ_0 containing $\mathcal{D}_0$ such that if some $\mathcal{D}'_0 \in \Xi$ and $\mathcal{D}''_0 \in C(\mathcal{D})$ share some compact edge, then $\mathcal{D}'_0 \in C(\mathcal{D})$. Obviously, all elements of $C(\mathcal{D})$ must have the same type of decomposition. Let $C^j(\mathcal{D})$ be the polyhedral complex consisting of all $\mathcal{D}_0'^j$ with $D'_0 \in C(\mathcal{D})$. If for example $\mathcal{D}_0$ decomposes as in (i), then $C^0(\mathcal{D})$ is simply a translate of $C(\mathcal{D})$.

For the remainder of this section we will assume that Ξ_0 is complete, that is, that $|\Xi_0| = N_{\mathbb{Q}}$. Note that for $\mathcal{D} \in \Xi$, the boundary of the polyhedral complex $C(\mathcal{D})$ has a rather special shape. Indeed, consider any vertex v on the boundary of $C(\mathcal{D})$. Then all faces of the boundary of $C(\mathcal{D})$ containing v must in fact be shifted cones with origin v. If $v_1, \ldots, v_k$ are all the vertices on the boundary of $C(\mathcal{D})$, then $N_{\mathbb{Q}} \setminus |C(\mathcal{D})|$ has exactly k connected components, each containing exactly one vertex v_i in its closure; for an example see figure 3, where each connected component has a different shade of gray.

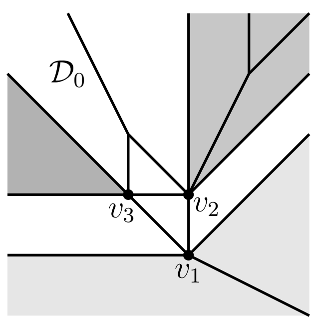


Figure 3: Connected components of $N_{\mathbb{Q}} \setminus |C(\mathcal{D})|$

Note that even for nontrivial deformations, the Minkowski decomposition of Ξ_0 induces a decomposition of its vertices: for each vertex $v \in \Xi_0$, we have a decomposition $v = v^0 + v^1$. Consider any sequence of vertices $\gamma = (v_0, v_1, \ldots, v_m)$, $v_i \in \Xi_0$, such that v_i, v_{i+1} are either the endpoints of an edge in Ξ_0 or $v_i = v_{i+1}$; we call γ a path from v_0 to v_m. Such a path γ induces paths γ^j, $j = 0, 1$ in the polyhedral complexes $\{\mathcal{D}_0^j\}_{\mathcal{D} \in \Xi}$. Indeed, set $\gamma^j = (v_0^j, v_1^j, \ldots, v_m^j)$. We call a path reduced if no consecutive vertices are equal; note that any path has a unique reduction. A path is called trivial if its reduction consists of a single vertex. A path γ from u to v and path γ' from v to w can be composed in a natural manner.

Lemma 2.8. *Consider vertices $v \neq w$ of Ξ_0 such that $v^j = w^j$ for some $j \in \{0,1\}$. Then there is a path γ from v to w such that the induced path γ^j is trivial.*

Proof. For simplicity assume that $j = 0$. Consider any path γ' from v to w. We can shorten γ' to a reduced path $\gamma = (v = v_0, \ldots, v_m = w)$ from v to w where we remove any loops, that is, we eliminate all vertices between any vertex appearing multiple times. We will show that γ^0 is in fact trivial.

For each $0 \leq i \leq m-1$, let $C_i = C(\mathcal{D})$ for some $\mathcal{D}$ with v_i and v_{i+1} vertices of $\mathcal{D}_0$. Note that C_i doesn't depend on the choice of such $\mathcal{D}$ due to the definition of $C(\mathcal{D})$. Let $(i_k)_k$ be the subsequence of $0, \ldots, m-1$ consisting of all i_k such that $C_{i_k} \neq C_{i_k - 1}$. Then one easily checks that all C_{i_k} are distinct. Furthermore, we can uniquely write γ as a composition of reduced nontrivial paths γ_{i_k}, where the vertices of γ_{i_k} are vertices in C_{i_k}.

Now, suppose that the elements of some C_{i_k} have a decomposition of type (ii) above. Then one easily checks that $\gamma^0_{i_k}$ is automatically trivial. In particular, if the elements of all C_{i_k} have a decomposition of type (ii), then γ^0 is trivial as desired.

Suppose instead that there is some k with C_{i_k} having a decomposition of type (i). We claim that this cannot be. Indeed, let k be the smallest such number. Then γ_{i_k} is a nontrivial path from v_{i_k} to $v_{i_{k+1}}$, and $\gamma^0_{i_k}$ is a nontrivial path from $v^0_{i_k}$ to $v^0_{i_{k+1}} = v^0_{i_k} + v_{i_{k+1}} - v_{i_k}$. Now, v_{i_k} and $v_{i_{k+1}}$ are vertices on the boundary of C_{i_k} and thus correspond to different connected components of $N_{\mathbb{Q}} \setminus |C_{i_k}|$, or likewise, to different connected components of $N_{\mathbb{Q}} \setminus |C^0_{i_k}|$. Now, one easily checks that the only possible vertex in $C^0_{i_l} \cap C^0_{i_k}$ for $l > k$ is $v^0_{i_{k+1}}$, if any, since all $C^0_{i_l}$, $l > k$ are contained in the closure of the connected component of $N_{\mathbb{Q}} \setminus |C^0_{i_k}|$ corresponding to $v^0_{i_{k+1}}$. Furthermore, the paths $\gamma^0_{i_l}$ and in particular the vertex $v^0_m = w^0 = v^0$ are all in this connected component. But clearly v^0 is also in the connected component corresponding to $v^0_{i_k}$, a contradiction. Thus, this case cannot occur, and γ^0 is the trivial path as desired. ☐

Remark. In the above proof, we used that Ξ_0 is complete, and that π is locally trivial in the description of the $C(\mathcal{D})$. We believe that the statement will still hold if $|\Xi_0|$ is convex and π arbitrary, but have yet to find a proof. If $|\Xi_0|$ isn't convex, there are easy counterexamples.

Now consider any support function $f \in \mathrm{SF}(\Xi)$. For any reduced path $\gamma = (v_0, v_1, \ldots, v_m)$ in Ξ_0 and $j = 0, 1$, we define

$$f_0(\gamma, j) = \sum_{i=1}^{m} \frac{|v_i^j - v_{i-1}^j|}{|v_i - v_{i-1}|} \Big(f_0(v_i) - f_0(v_{i-1}) \Big).$$

Note that in the above equation, the coefficients $\frac{|v_i^j - v_{i-1}^j|}{|v_i - v_{i-1}|}$ are always either 0 or 1.

Lemma 2.9. *Consider vertices v, w of Ξ_0 and two reduced paths γ, γ' from v to w. For any support function $f \in \mathrm{SF}(\Xi)$, we have that*

$$f_0(\gamma, j) = f_0(\gamma', j)$$

for $j = 0, 1$.

Proof. If γ is the composition of γ_1 and γ_2, then one has that $f_0(\gamma, j) = f_0(\gamma_1, j) + f_0(\gamma_2, j)$. Thus, it is sufficient to show that $f_0(\gamma, j) = 0$ for any reduced path γ with equal start and end points, and no other repeated vertices; we call such a path closed. Now, one easily sees that this is the case if all vertices of a closed path γ lie in some $C(\mathcal{D})$ for $\mathcal{D} \in \Xi$. On the other hand, it is not difficult to check that the vertices of any closed path must in fact lie in a complex of the form $C(\mathcal{D})$. $\square$

We now turn to the proof of the surjectivity of $\pi_{s,0}$:

Proof of theorem 2.4. Let Ξ_0 be complete and suppose π is locally trivial. Consider some element $f \in \mathrm{CaSF}'(\Xi)$. We show how to construct a support function $h^{(s)} \in \mathrm{CaSF}'(\Xi^{(s)})$ with $h^{(0)} = f$, from which the first part of the theorem then follows. Note that to define a continuous piecewise affine function on any polyhedral subdivision, it suffices to know its linear part and its values on the vertices of the subdivision.

Now fix some vertex $b \in \Xi_0^{(0)}$. Consider any vertex $w \in \Xi_0^{(s)}$. For any path γ from b to v with $v^0 = w$, define

$$h_0^{(s)}(w) = f_0(b) + f_0(\gamma, 0).$$

This definition depends on neither v nor γ. Indeed, for fixed v, independence of the path γ follows from lemma 2.9. On the other hand consider vertices $v \neq v'$ with $v^0 = v'^0 = w$, and paths γ_v and $\gamma_{v'}$ from b to v respectively v'. Then there is a path $\gamma_{v,v'}$ from v to v' such that $\gamma_{v,v'}^0$ is trivial by lemma 2.8. Then $f_0(\gamma_{v'}, 0) = f_0(\gamma_v, 0) + f_0(\gamma_{v,v'}, 0) = f_0(\gamma_{v'}, 0)$, and so $h_0^{(s)}(w)$ doesn't depend on the point v, either. For a vertex $w \in \Xi_s^{(s)}$ we can similarly define $h_s^{(s)}(w) = f_0(\gamma, 1)$ for any path γ from $b = v_0$ to v_m with $v_m^0 = w$.

Now, we can make $h_0^{(s)}$ and $h_s^{(s)}$ into continuous piecewise affine functions by requiring that both have the same linear part as f; one easily checks that this in fact yields uniquely defined continuous piecewise affine functions. Now set $h_P^{(s)} = f_P$ for $P \neq 0, s$. Setting $h^{(s)} = \sum_{P \in \mathbb{P}^1} h_P^{(s)} \cdot P$, it immediately follows from the construction that $h^{(s)} \in \mathrm{CaSF}'(\Xi^{(s)})$ and that $h^{(0)} = f$, thus proving the first part of the theorem.

Now assume that $\operatorname{rank} \operatorname{Pic}'(X_s) = \operatorname{rank} \operatorname{Pic}'(X_0)$. Then the map $\operatorname{Pic}'(X_s) \otimes \mathbb{Q} \to \operatorname{Pic}'(X_0) \otimes \mathbb{Q}$ induced by $\pi_{s,0}$ is an isomorphism. However, one easily checks that $\operatorname{Pic}'(X_0)$ is torsion free, since multiples of a non-constant support function are still non-constant, and $\operatorname{Pic}(\mathbb{P}^1)$ is torsion free. Thus, given any support function $f \in \mathrm{CaSF}'(\Xi^{(0)})$, we can find a not necessarily integral support function $\tilde{h}^{(s)} \in \mathrm{CaSF}'_{\mathbb{Q}}(\Xi^{(s)})$ with $\tilde{h}^{(0)} = f$, where $\mathrm{CaSF}_{\mathbb{Q}}$ is defined as CaSF without the integral condition. However, it is clear from the construction of $\tilde{h}^{(0)}$ from $\tilde{h}^{(s)}$ that $\tilde{h}^{(s)}$ must have integral slopes. Now consider any lattice point $v \in |\Xi_0|$ with not-necessarily unique decomposition $v = v^0 + v^1$, with v^j lattice points in $\mathcal{D}_0^j$ for some common $\mathcal{D} \in \Xi$. Set $h_0^{(s)} = \tilde{h}_0^{(s)} - h_0^{(s)}(v^0)$, $h_s^{(s)} = \tilde{h}_s^{(s)} + h_0^{(s)}(v^0)$, and $h_P^{(s)} = \tilde{h}_P^{(s)}$ for $P \neq 0, s$. Then one easily checks that $h^{(s)} = \sum h_P^{(s)} \cdot P$ is an element of $\mathrm{CaSF}'(\Xi^{(s)})$ with $h^{(0)} = f$. $\square$

3 Rational $\mathbb{C}^*$-Surfaces

3.1 Multidivisors, Weighted Graphs, and Continued Fractions

In general, a T-variety is not determined by the slices Ξ_y of its divisorial fan Ξ on Y. However, for complete $\mathbb{C}^*$-surfaces only a small bit of additional information is needed; we encode the slices as well as this additional information in a *multidivisor*.

Definition. A *multidivisor* $\mathcal{M}$ on a smooth projective curve Y a collection of polyhedral proper subdivisions $\mathcal{M}_P$ of $\mathbb{Q}$ for each $P \in Y$ together with $(\mathcal{M}_-, \mathcal{M}_+) \in \{\circ, \bullet\}^2$ such that:

(i) Only finitely many $\mathcal{M}_P$ differ from the subdivision induced by a single vertex at 0;

(ii) If $\mathcal{M}_- = \circ$, then
$$\sum_{P \in Y} \min_{v \in \mathcal{M}_P \text{vertex}} v < 0;$$

(iii) If $\mathcal{M}_+ = \circ$, then
$$\sum_{P \in Y} \max_{v \in \mathcal{M}_P \text{vertex}} v > 0.$$

If Ξ is a divisorial fan on Y with $N = \mathbb{Z}$, we get a multidivisor by setting $\mathcal{M}_P = \Xi_P$ and setting $M_- = \circ$ exactly when there is $\mathcal{D} \in \Xi$ with $\mathrm{tail}(\mathcal{D}) = \mathbb{Q}_{\leq 0}$ and $\mathrm{Loc}(\mathcal{D}) = Y$, and likewise setting $M_+ = \circ$ when there is $\mathcal{D} \in \Xi$ with $\mathrm{tail}(\mathcal{D}) = \mathbb{Q}_{\geq 0}$ and $\mathrm{Loc}(\mathcal{D}) = Y$. On the other hand, for any multidivisor $\mathcal{M}$, we can find a compatible divisorial fan Ξ, and the resulting T-variety only depends on the multidivisor, see [Süß08]. We thus will speak of $X(\mathcal{M})$ for $\mathcal{M}$ a multidivisor. Note that $X(\mathcal{M})$ is a toric surface if and only if $Y = \mathbb{P}^1$ and $\mathcal{M}$ has at most two non-trivial slices. The process of going from a fan defining a toric surface to a corresponding multidivisor is briefly explained in the proof of theorem 3.4 and can be found in more detail in [PS08], remark 2.8. There is also an easy criterion for the smoothness of $X(\mathcal{M})$, see [Süß08].

Remark. For any multidivisor $\mathcal{M}$, the T-invariant prime Weil divisors of $X(\mathcal{M})$ correspond either to vertices $v \in \mathcal{M}_P$, or values $\bullet$ of $\mathcal{M}_-, \mathcal{M}_+$, see [PS08]. We denote these divisors by respectively $D_{v,P}$ or by D_-, D_+. If $X(\mathcal{M})$ is smooth, Weil divisors and Cartier divisors are equivalent, so we can use them interchangeably. For any vertex $v \in \mathcal{M}_P$, denote by $\lambda(v)$ the denominator of v when written in lowest terms; we call this the *height* of v.

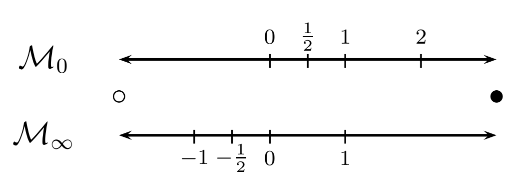


Figure 4: A possible multidivisor

Example. A multidivisor $\mathcal{M}$ can be presented quite nicely as a picture, by showing all nontrivial slices and noting $\mathcal{M}_-$ and $\mathcal{M}_+$ on the left and right sides of the picture. For example, consider $\mathbb{P}^2$ as a toric variety, blown up in two of the three toric fixpoints, with further blowups in the four resulting fixpoints of the exceptional divisors. A multidivisor yielding this toric variety is pictured in figure 4.

In general, complete $\mathbb{C}^*$-surfaces can also be described in terms of weighted graphs, see [OW77]. If the surface is in fact a toric variety, then this graph is circular, see [Ful93]. Up to isomorphism we can thus represent a smooth, complete toric surface X by some sequence $(b_0, \dots, b_l)$, where the $-b_i$ are the self intersection numbers of the torus invariant prime divisors ordered in a suitable manner. In this case, we simply write $X = \mathrm{TV}(b_0, \dots, b_l)$. Alternatively, if $\Sigma \subset \mathbb{Z}^2 \otimes \mathbb{Q}$ is some fan such such that X is the associated toric surface, we write $X = \mathrm{TV}(\Sigma)$. We then denote by ρ_i the ray of Σ corresponding to b_i. By abuse of notation we denote the minimal generator of the ray ρ_i by ρ_i as well. Note then that $b_i \cdot \rho_i = \rho_{i-1} + \rho_{i+1}$ and that the Picard number $\rho(X)$ of X is $l-1$. Furthermore, one has has the equality $\sum b_i = 3l - 9$.

Let $c_1, c_2, \dots, c_k \in \mathbb{Z}$. The continued fraction $[c_1, c_2, \dots, c_k]$ is inductively defined as follows if no division by 0 occurs: $[c_k] = c_k$, $[c_1, c_2, \dots, c_k] = c_1 - 1/[c_2, \dots, c_k]$. Now consider some $l+1$ tuple $(b_0, \dots, b_l)$ defining a smooth toric surface, and suppose that $b_0 < 0$ and $l > 2$. Using induction on l, one can easily show that there exists a unique index α, $1 < \alpha < l$ such that $[b_1, \dots, b_{\alpha-1}]$ is well defined and equals zero, or equivalently, that $\rho_\alpha = -\rho_0$. If we are in this situation, we define

$$\gamma = \sum_{i=1}^{\alpha-1} (3 - b_i) - 3. \tag{3}$$

We will use the following lemma in the next section:

Lemma 3.1. *We always have $\gamma \geq 0$. Likewise, $b_0 + b_\alpha - \gamma \geq 0$. Finally, for $R \in (\mathbb{Z}^2)^*$ such that $\langle \rho_\alpha, R \rangle = 1$, we have $\langle \rho_{\alpha-1}, R \rangle - \langle \rho_1, R \rangle = \gamma$.*

Proof. All statements can be easily shown by induction on l. □

3.2 Deformations of Rational $\mathbb{C}^*$-Surfaces

From now on we will concentrate on smooth, complete, rational $\mathbb{C}^*$-surfaces, which we will simply call rational $\mathbb{C}^*$-surfaces. Let $\pi : X^{\mathrm{tot}} \to S$ be a homogeneous one-parameter deformation of the rational $\mathbb{C}^*$-surface X_0. For any $s \in S^*$, we say that X_0 *deforms* to X_s and write $X_0 \rightsquigarrow X_s$. Conversely, we say that X_s *degenerates* to X_0. By an abuse of terminology we will call X_s the general fiber. We first describe a nice way of encoding a degeneration from some X_s to X_0.

Definition. Let $\mathcal{M}$ be a multidivisor giving a rational $\mathbb{C}^*$-surface and choose some $s \in \mathbb{P}^1$, $s \neq 0$. A *degeneration diagram* for the multidivisor $\mathcal{M}$ consists of the pair $(\mathcal{M}, G)$, where G is a connected graph on the vertices of $\mathcal{M}_0$ and $\mathcal{M}_s$ such that:

(i) G is bipartite with respect to the natural partition induced by $\mathcal{M}_0$ and $\mathcal{M}_s$;

(ii) G can be realized in the plane with all edges being line segments by embedding $\mathcal{M}_0$ and $\mathcal{M}_s$ in parallel lines;

(iii) Every vertex of G with degree strictly larger than one is a lattice point.

To a degeneration diagram $(\mathcal{M}, G)$ we can associate a deformation π as follows. Let $\mathcal{M}^{(0)}$ be the multidivisor with $\mathcal{M}_\pm^{(0)} = \mathcal{M}_\pm$, $\mathcal{M}_P^{(0)} = \mathcal{M}_P$ for $P \in \mathbb{P}^1 \setminus \{0, s\}$, $\mathcal{M}_s^{(0)}$ trivial, and $\mathcal{M}_0^{(0)}$ the subdivision of $\mathbb{Q}$ with vertices of the form $v_0 + v_s$, with $\overline{v_0 v_s}$ an edge of G. Let $\Xi^{(0)}$ be a divisorial fan compatible with $\mathcal{M}^{(0)}$. Each polyhedron $[v^-, v^+]$ in the subdivision $\Xi_0^{(0)}$ comes with a natural decomposition $[v^-, v^+] = [v_0^-, v_0^+] + [v_s^-, v_s^+]$, where $v_P^j \in \mathcal{M}_P$ and $\overline{v_0^j v_s^j}$ is an edge of G. This gives an admissible decomposition of Ξ_0 and thus a deformation $\pi : X^{\mathrm{tot}} \to S$ of $X(\mathcal{M}^{(0)})$, with $X_s = X(\mathcal{M})$. Conversely, any homogeneous one-parameter deformation of a rational $\mathbb{C}^*$-surface with fiber $X_s = X(\mathcal{M})$ arises from a degeneration diagram of the form $(\mathcal{M}, G)$.

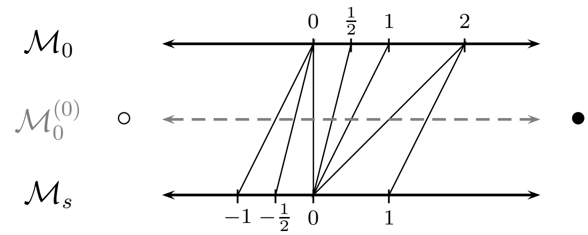


Figure 5: A possible degeneration diagram

Example. Similar to multidivisors, degeneration diagrams can be presented easily in picture form. Indeed, for a degeneration diagram $(\mathcal{M}, G)$, we draw the multidivisor $\mathcal{M}$ as described in the previous section and then draw line segments between vertices of $\mathcal{M}_0$ and $\mathcal{M}_s$ connected by an edge of G. For example, figure 5 presents a degeneration diagram for the multidivisor $\mathcal{M}$ of figure 4 with $s = \infty$. The resulting slice $\mathcal{M}_0^{(0)}$ can then be seen quite easily as the induced subdivision on the dashed line in between $\mathcal{M}_0$ and $\mathcal{M}_s$ scaled by a factor of two.

To distinguish between Weil divisors on X_0 and X_s, we write them with a superscript, i.e. $D_{v,P}^{(0)}$ and $D_{v,P}^{(s)}$, etc. We shall now see how such deformations are compatible with blowing up and blowing down. We will need the following lemma:

Lemma 3.2. *Let $(\mathcal{M}, G)$ be a degeneration diagram. For any edge $\overline{v_0 v_s}$ of G, $v_0 \in \mathcal{M}_0, v_s \in M_s$, with $(D_{v_0+v_s,0}^{(0)})^2 = -1$, one of the vertices v_0, v_s must have degree one. If both vertices have degree one, the deformation corresponding to $(\mathcal{M}, G)$ is trivial.*

Proof. If $v_0 + v_s$ lies to the left or to the right of all other vertices of $\mathcal{M}_0^{(0)}$, then the edge $\overline{v_0 v_s}$ lies to the left or right of all other edges of G as well; it is then clear that either v_0 or v_s must have degree one. If on the other hand $v_0 + v_s$ has left and right neighboring vertices $v', v'' \in \mathcal{M}_0^{(0)}$, then $\lambda(v_0 + v_s) = \lambda(v') + \lambda(v'') > 1$. If neither v_0 nor v_s has degree one, they must both be lattice points, in which case $\lambda(v_0 + v_s) = 1$, a contradiction.

The second claim follows easily from the observation that if both vertices have degree one, G can only have one edge. $\square$

Using this lemma, it is clear how to *blow down* any deformation. Indeed, let π correspond to the degeneration diagram $(\mathcal{M}, G)$, and let $\phi : X_0 \to X_0'$ be the contraction of an invariant minus one curve.

Suppose first of all that this curve is of the form $D_{v,P}^{(0)}$ for $P \neq 0$ or of the form $\mathcal{D}_\pm$. Then we get a new multidivisor $\mathcal{M}'$ by respectively removing the vertex v from the subdivision $\mathcal{M}_P$ or by setting $\mathcal{M}'_-$ or $\mathcal{M}'_+$ equal to $\circ$. Setting $G' = G$, we then have that $(\mathcal{M}', G)$ is a degeneration diagram with $X(\mathcal{M}'^{(0)}) = X_0'$ and with $X_s' = X(\mathcal{M}')$ a blowdown of X_s.

On the other hand, suppose that ϕ blows down a curve of the form $D_{v,0}^{(0)}$. Then v corresponds to an edge $\overline{v_0 v_s}$ of G and by the above lemma, either v_0 or v_s must have degree one; assume without loss of generality that this is v_0. We then get a new multidivisor $\mathcal{M}'$ by removing the vertex v_0 from the subdivision $\mathcal{M}_0$. Furthermore, we have a graph G' on $\mathcal{M}'$ attained from G by removing the edge $\overline{v_0 v_s}$. Due to the fact that v_0 had degree one in G, one easily checks that $(\mathcal{M}', G')$ is a degeneration diagram.

As in the other case, we have $X(\mathcal{M}'^{(0)}) = X_0'$ and $X_s' = X(\mathcal{M}')$ a blowdown of X_s. In this manner we define the *blowdown of* $(\mathcal{M}, G)$ *by* ϕ to be $(\mathcal{M}', G')$. We call the deformation corresponding to $(\mathcal{M}', G')$ the blowdown of π by ϕ.

It is also possible to lift a deformation $\pi : X \to S$ by an invariant *blowup* ϕ of either the special fiber X_0 or the fiber X_s. Indeed, let $(\mathcal{M}, G)$ be the corresponding degeneration diagram.

The first possible type of blowup of X_0 or X_s is by blowing up in an elliptic fixpoint of the $\mathbb{C}^*$ action, that is, by replacing a $\circ$ value of $\mathcal{M}_\pm^{(0)}$ or $\mathcal{M}_\pm$ by $\bullet$. If we define $\mathcal{M}'$ to be equal to $\mathcal{M}$ with the relevant modification of $\mathcal{M}_-$ or $\mathcal{M}_+$ and set $G' = G$, we get a degeneration diagram $(\mathcal{M}', G')$ with either $X(\mathcal{M}'^{(0)})$ or $X(\mathcal{M}')$ the desired blowup of X_0 or respectively X_s.

Suppose instead that the blowup of X_0 or X_s corresponds to inserting a vertex v in the subdivision $\mathcal{M}_P^{(0)} = \mathcal{M}_P$ for $P \neq 0, s$. Then if we define $\mathcal{M}'$ to come from $\mathcal{M}$ by adding the vertex v to $\mathcal{M}_P$ and setting $G = G'$, we get a degeneration diagram $(\mathcal{M}', G')$ with the same property as in the previous case.

Suppose now that a blowup of X_0 corresponds to inserting a vertex v in the subdivision $\mathcal{M}_0^{(0)}$. This corresponds to the insertion of a vertex $\tilde{v}$ in either $\mathcal{M}_0$ or $\mathcal{M}_s$, which in turn corresponds to a blowup of X_s.[3] So assume that we have a blowup of X_s of this form. Then we can define a multidivisor $\mathcal{M}'$ from $\mathcal{M}$ similar to the previous cases. Likewise, we can define a graph G' on the vertices of $\mathcal{M}_0', \mathcal{M}_s'$ by adding an edge between $\tilde{v}$ and the unique vertex connected to both all neighboring vertices of $\tilde{v}$.

This defines a degeneration diagram $(\mathcal{M}', G')$ with the same property as above. In such cases, we call $(\mathcal{M}', G')$ a *blowup of* $(\mathcal{M}, G)$ *by* ϕ. We can sum up the preceding constructions in the following proposition:

Proposition 3.3. *Let $(\mathcal{M}, G)$ be a degeneration diagram with corresponding special fiber X_0 and general fiber X_s.*

[3]Note that the placement of the vertex in either $\mathcal{M}_0$ or $\mathcal{M}_s$ is uniquely determined if v isn't an extremal vertex of $\mathcal{M}_0^{(0)}$.

(i) If $\phi : X_0 \to X'_0$ is a blowdown of an invariant curve, there is a unique degeneration diagram $(\mathcal{M}', G')$ called the blowdown of $(\mathcal{M}, G)$ by ϕ such that $X(\mathcal{M}'^{(0)}) = X'_0$ and $X(\mathcal{M}')$ is an invariant blowdown of X_s.

(ii) if $\phi : X'_0 \to X_0$ is an invariant blowup, there is a degeneration diagram $(\mathcal{M}', G')$ called a blowup of $(\mathcal{M}, G)$ by ϕ such that $X(\mathcal{M}'^{(0)}) = X'_0$ and $X(\mathcal{M}')$ is an invariant blowup of X_s.

(iii) if $\phi : X'_s \to X_s$ is an invariant blowup, there is a unique degeneration diagram $(\mathcal{M}', G')$ called the blowup of $(\mathcal{M}, G)$ by ϕ such that $X(\mathcal{M}') = X'_s$ and $X(\mathcal{M}'^{(0)})$ is an invariant blowup of X_0.

In section 3.4 we will see that these constructions commute with the corresponding maps of divisors. To end this section, we shortly turn our attention to homogeneous deformations where all fibers are toric. The following theorem tells us how a number of these can be described nicely in terms of self-intersection numbers:

Theorem 3.4. *Consider a smooth complete toric surface $X_0 = \mathrm{TV}(b_0, \ldots, b_l)$ such that $b_0 < 0$ and $l > 2$. Then there exists a homogeneous deformation of X_0 with toric general fiber*

$$X_s = \mathrm{TV}(b_0 + \gamma + 2r, b_{\alpha-1}, \ldots, b_1, b_\alpha - \gamma - 2r, b_{\alpha+1}, \ldots, b_l),$$

where α and γ are as defined in the previous section and $0 \le r \le -b_0$.

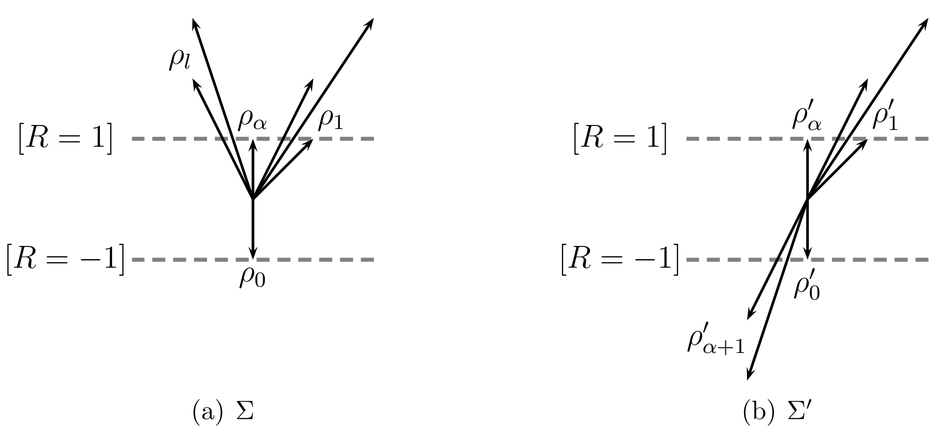


Figure 6: Possible fans from the proof of theorem 3.4

Proof. Let $X_0 = \mathrm{TV}(\Sigma)$ for some fan $\Sigma \subset \mathbb{Z}^2 \otimes \mathbb{Q}$ with rays ρ_i corresponding to the numbers b_i, see for example figure 6(a). Consider the unique $R \in (\mathbb{Z}^2)^*$ such that $\langle \rho_0, R \rangle = -1$ and $\langle \rho_1, R \rangle = r$. We transform the fan Σ into a multidivisor $\mathcal{M}^{(0)}$ by taking $\mathcal{M}_0^{(0)}$ to be the subdivision induced by Σ on the line $[R = 1]$, with ρ_α set as the origin, and by setting $\mathcal{M}_-^{(0)}, \mathcal{M}_+^{(0)}$ to be $\bullet$ if and only if $r = 0$ or $r = -b_0$, respectively. Note that the condition on $0 \le r \le -b_0$ is exactly such that Σ subdivides the line $[R = -1]$ only with the ray ρ_0. Thus, we have $X_0 = X(\mathcal{M}^{(0)})$; for further details see [PS08], remark 2.8.

We now construct a degeneration diagram $(\mathcal{M}, G)$ giving special fiber X_0. Indeed, let the vertices of $\mathcal{M}_0$ consist of those vertices $v \in \mathcal{M}_0^{(0)}$ with $v \geq 0$. Likewise, let the vertices of $\mathcal{M}_s$ consist of those vertices $v \in \mathcal{M}_0^{(0)}$ with $v \leq 0$. Furthermore, set $\mathcal{M}_- = \mathcal{M}_-^{(0)}$ and $\mathcal{M}_+ = \mathcal{M}_+^{(0)}$. We then define G to be the graph having edges $\overline{v_0 v_s}$ with $v_0 \in \mathcal{M}_0, v_s \in \mathcal{M}_s$ and either $v_0 = 0$ or $v_s = 0$. One easily confirms that $(\mathcal{M}, G)$ is indeed a degeneration diagram with $\mathcal{M}_0^{(0)}$ being the multidivisor for the corresponding special fiber.

Now, we see that the general fiber $X_s = X(\mathcal{M})$ is toric, since $\mathcal{M}$ only has two nontrivial slices. In fact, by embedding $\mathcal{M}_0$ and $\mathcal{M}_s$ in height one and minus one respectively, we recover a fan Σ' with $X_s = X(\mathcal{M}) = X(\Sigma')$, see for example figure 6(b). Σ' then has rays $\rho'_0, \ldots, \rho'_l$ ordered cyclically with $\rho'_i = \rho_i$ for $\alpha \leq i \leq l$ or $i = 0$ and ρ'_i a vertical reflection for $0 < i < \alpha$. Let $-b'_i$ be the self-intersection number of the divisor corresponding to the ρ'_i; then X_s is represented by the chain $(b'_0, \ldots, b'_l)$. Now, it is immediate from this description that $b'_i = b_{\alpha-i}$ for $1 \leq i < \alpha$ and that $b'_i = b_i$ for $\alpha < i \leq l$. Furthermore,

$$b'_\alpha = \langle \rho'_{\alpha-1}, R \rangle + \langle \rho'_{\alpha+1}, R \rangle = -\langle \rho_1, R \rangle + \langle \rho_{\alpha+1}, R \rangle. \tag{4}$$

We also have $\langle \rho_{\alpha-1}, R \rangle = r + \gamma$ by lemma 3.1. We can then rewrite equation (4) as

$$b'_\alpha = -r + b_\alpha - \langle \rho_{\alpha-1}, R \rangle = b_\alpha - \gamma - 2r.$$

Since the sum of all the intersection numbers must remain constant, we also have $b'_0 = b_0 + \gamma + 2r$, completing the proof.

□

3.3 Deformation Connectedness

Let X and X' be two rational $\mathbb{C}^*$-surfaces.

Definition. We say that X and X' are *homogeneously deformation connected* if there is a finite sequence $X = X^0, X^1, \ldots, X^k = X'$ with $X^i \rightsquigarrow X^{i-1}$ or $X^{i-1} \rightsquigarrow X^i$ for all $1 \leq i \leq k$.

It is well-known that a Hirzebruch surface of even parity cannot be deformed to a Hirzebruch surface of odd parity and vice versa. An obstruction to such a deformation can be found by comparing the Chow rings. If we instead consider rational surfaces of fixed Picard number $\rho > 2$, it is an easy exercise to see that all the Chow rings are isomorphic. Thus, the obstruction to deformation we had for the case $\rho = 2$ no longer exists. In fact, for rational $\mathbb{C}^*$-surfaces it is sufficient to consider homogeneous deformations:

Theorem 3.5. *Consider the set of all rational $\mathbb{C}^*$-surfaces with Picard number ρ for any integer $\rho > 2$. All elements of this set are homogeneously deformation connected.*

The proof of this theorem will constitute the remainder of this section. We first prove the following lemma:

Lemma 3.6. *Any rational $\mathbb{C}^*$-surface X is connected through a sequence of homogeneous degenerations to a toric surface.*

Proof. Let $\mathcal{M}$ be a multidivisor with $X = X(\mathcal{M})$. Suppose that $\mathcal{M}$ has more than three non-trivial slices. Then there are non-trivial slices $\mathcal{M}_P, \mathcal{M}_Q$ with $P \neq Q$ such that the left-most vertex v_P of $\mathcal{M}_P$ and the right-most vertex v_Q of $\mathcal{M}_Q$ are lattice points; this follows from the smoothness criterion of [Süß08]. Setting $0 = P, s = Q$ and considering the graph G on the vertices of $\mathcal{M}_P, \mathcal{M}_Q$ with edges of the form $\overline{v_0 v_Q}$ and $\overline{v_s v_P}$ for $v_0 \in \mathcal{M}_P$, $v_s \in \mathcal{M}_Q$ gives a degeneration diagram $(\mathcal{M}, G)$. The corresponding special fiber has one less non-trivial slice than X.

We can apply the above procedure inductively, and can thus assume that $\mathcal{M}$ has at most three non-trivial slices. If $\mathcal{M}$ has less than three non-trivial slices, then $X(\mathcal{M})$ is toric, and we are done. If as above there are non-trivial slices $\mathcal{M}_P, \mathcal{M}_Q$ with $P \neq Q$ such that the left-most vertex v_P of $\mathcal{M}_P$ and the right-most vertex v_Q of $\mathcal{M}_Q$ are lattice points, then we can once again proceed as above and degenerate to something with only two non-trivial slices. We thus must only consider the remaining case, which is that where $\mathcal{M}$ has three non-trivial slices $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_\infty$ and $\mathcal{M}_0, \mathcal{M}_\infty$ have no extremal lattice vertices and both extremal vertices of $\mathcal{M}_1$ are lattice points. We show that this is actually impossible.

In this case, we can actually assume that the left-most vertex of $\mathcal{M}_1$ is 0, and that the right-most vertex is n. Let u_0^l/v_0^l, u_∞^l/v_∞^l be the left-most vertices of $\mathcal{M}_0$ and $\mathcal{M}_\infty$ written in lowest terms and let u_0^r/v_0^r, u_∞^r/v_∞^r similarly be the right-most vertices. Due to smoothness we have

$$-u_0^l v_\infty^l - u_\infty^l v_0^l = 1; \tag{5}$$

$$u_0^r v_\infty^r + u_\infty^r v_0^r + v_0^r v_\infty^r \cdot n = 1. \tag{6}$$

Furthermore, we of course have

$$u_P^l v_P^r \leq u_P^r v_P^l \tag{7}$$

for $P = 0, \infty$. Solving equations (5) and (6) for v_0^l and v_0^r, substituting for these expressions in (7) for $P = 0$, and rearranging terms gives us

$$v_0^r v_\infty^r + v_0^l v_\infty^l + u_\infty^l v_\infty^r v_0^r v_0^l \geq u_\infty^r v_\infty^l v_0^l v_0^r + v_0^l v_\infty^l v_0^r v_\infty^r n.$$

Combining this with (7) for $P = \infty$ then gives us

$$v_0^r v_\infty^r + v_0^l v_\infty^l \geq v_0^l v_\infty^l v_0^r v_\infty^r n.$$

This however is a contradiction, since $n \geq 1$ and $v_0^l, v_\infty^l, v_0^r, v_\infty^r \geq 2$. Thus, this case never arises and we can always degenerate to a toric surface. □

In general, one can always construct a rational surface by iteratively blowing up a Hirzebruch surface in a number of points. This can be done equivariantly for rational $\mathbb{C}^*$-surfaces. For multidivisors $\mathcal{M}$ with $\mathcal{M}_- = \mathcal{M}_+ = \bullet$, this is stated in [OW77]. However, we know of no proof of the general case and thus provide one here as an easy corollary of the above lemma:

Corollary 3.7. *Any rational $\mathbb{C}^*$-surface X with Picard number larger than two can be constructed from a Hirzebruch surface by a series of equivariant blowups.*

Proof. Suppose that $X = X_s$ isn't a Hirzebruch surface. By lemma 3.6, we know that X degenerates to some toric variety X_0. But there is an invariant minus one curve on X_0 which can be blown down, since X_0 is toric, see [Ful93]. Blowing down the deformations from X_0 to X_s as in proposition 3.3 gives us a new general fiber X'_s which is an invariant blowdown of X_s. The proof then follows by induction on the Picard number. □

We will collect several more lemmata we shall need:

Lemma 3.8. *Consider a smooth fan Σ with rays $\rho_0, \dots, \rho_l$. Let Σ_1 and Σ_2 be the smooth fans attained by inserting a ray between ρ_0 and ρ_1 respectively ρ_1 and ρ_2. Then $\mathrm{TV}(\Sigma_1)$ is homogeneously deformation connected to $\mathrm{TV}(\Sigma_2)$.*

Proof. As in the proof of theorem 3.4, we can transform Σ into a multidivisor $\mathcal{M}$. For $P_1 \neq P_2 \in \mathbb{P}^1$, let $\mathcal{M}_{P_1}$ and $\mathcal{M}_{P_2}$ to be the subdivisions induced by Σ on the affine lines $[R = 1]$ and $[R = -1]$, where $R \in (\mathbb{Z}^2)^*$ is such that $\langle \rho_1, R\rangle = 0$ and $\langle \rho_0, R\rangle < 0$. We then set $\mathcal{M}_+ = \bullet$, and set $\mathcal{M}_- = \bullet$ only if there is $\alpha \neq 1$ such that $\langle \rho_\alpha, R\rangle = 0$. Then $X(\mathcal{M}) = X(\Sigma)$.

Now, for some $s \in \mathbb{P}^1 \setminus \{P_1, P_2\}$, let $\tilde{\mathcal{M}}$ be the multidivisor with $\tilde{\mathcal{M}}_P = \mathcal{M}_P$ for $P \neq s$, and $\tilde{\mathcal{M}}_s$ the subdivision of $\mathbb{Q}$ with vertices 0 and 1. For $i = 1, 2$, let G_i be the graph on the vertices of $\tilde{\mathcal{M}}_s$ and $\tilde{\mathcal{M}}_{P_i}$ with edges $\overline{v_s w}$ for either vertices $v_s = 0 \in \tilde{\mathcal{M}}_s$ and $w \in \tilde{\mathcal{M}}_{P_i}$ or vertices $v_s = 1 \in \tilde{\mathcal{M}}_s$ and w the right-most vertex in $\mathcal{M}_{P_i}$. Setting $P_i = 0$, one easily checks that $(\tilde{\mathcal{M}}, G_i)$ is a degeneration diagram with general fiber $X(\tilde{\mathcal{M}})$ and special fiber $\mathrm{TV}(\Sigma_i)$. Thus, we have homogeneous deformations from both $\mathrm{TV}(\Sigma_1)$ and $\mathrm{TV}(\Sigma_2)$ to some common rational $\mathbb{C}^*$-surface, making them deformation connected. □

Remark. The two deformations constructed in the above proof can be naturally glued together to give a flat family X^{tot} over $\mathbb{P}^1$ with fibers $X_0 = \mathrm{TV}(\Sigma_1)$ and $X_\infty = \mathrm{TV}(\Sigma_2)$. In this family, the fiber over any point $s \in \mathbb{P}^1$ is simply the blowup of $\mathrm{TV}(\Sigma)$ in s, where we have identified the base space $\mathbb{P}^1$ with the divisor corresponding to the ray ρ_1.

Lemma 3.9. *The set $\{\mathrm{TV}(b_0, 0, b_\alpha, 1, 1) \mid b_0 + b_\alpha = 3\}$ is homogeneously deformation connected.*

Proof. Consider b_0, b_α and b'_0, b'_α such that $b_0 + b_\alpha = b'_0 + b'_\alpha = 3$. Due to symmetry we can assume that b_0, b'_0 have the same parity. Let $b''_0 = -\max\{|b_0|, |b'_0|\}$. Then $\mathrm{TV}(b''_0, 0, n - b''_0, 1, 1)$ deforms to both $\mathrm{TV}(b_0, 0, b_\alpha, 1, 1)$ and to $\mathrm{TV}(b'_0, 0, b'_\alpha, 1, 1)$ by theorem 3.4, so the desired set is homogeneously deformation connected. □

We now turn to the proof of the theorem:

Proof of theorem 3.5. We will prove the theorem by induction on ρ. Suppose that $\rho = 3$. From lemma 3.6 we have that any rational $\mathbb{C}^*$-surface can be degenerated to a toric surface. Furthermore, one easily checks that every toric surface with Picard number 3 is of the form $\mathrm{TV}(b_0, 0, b_\alpha, 1, 1)$. Thus, for $\rho = 3$ the statement then follows from lemma 3.9.

Assume that the theorem holds for Picard number ρ, and consider any two rational $\mathbb{C}^*$-surfaces X^1, X^2 with Picard number $\rho + 1$. By again applying lemma 3.6, we can assume without loss of generality that X^1 and X^2 are toric. Let $\tilde{X}^i$ be an invariant blowdown of X^i. Then $\tilde{X}^1$ and $\tilde{X}^2$ are homogeneously deformation connected by the induction hypothesis,

and this series of deformations and degenerations can be blown up to connect $\hat{X}^1$ and $\hat{X}^2$, where $\hat{X}^i$ is an invariant blowup of $\tilde{X}^i$. Thus, we must only show that $\hat{X}^i$ and X^i are homogeneously deformation connected, that is, any two invariant blowups in a point of a common toric surface are homogeneously deformation connected. But this follows from repeated application of lemma 3.8, proving the theorem. □

3.4 Families of Divisors on $\mathbb{C}^*$-Surfaces

Suppose π is a one-parameter deformation of a rational $\mathbb{C}^*$-surface with $X_s = X(\mathcal{M})$ for some multidivisor $\mathcal{M}$. The map $\pi_{s,0} : \text{T-CaDiv}(X_s) \to \text{T-CaDiv}(X_0)$ can be described quite nicely in terms of Weil divisors and the corresponding deformation diagram $(\mathcal{M}, G)$. Indeed, the following proposition offers an explicit description:

Proposition 3.10. *For any $P \in S \setminus \{0, s\}$ and $v_P \in \mathcal{M}_P$, $v_0 \in \mathcal{M}_0$, $v_s \in \mathcal{M}_s$, the map $\pi_{s,0}$ is defined by:*

$$D_\pm^{(s)} \mapsto D_\pm^{(0)} \qquad\qquad D_{v_P,P}^{(s)} \mapsto D_{v_P,P}^{(0)}$$

$$D_{v_0,0}^{(s)} \mapsto \sum_{\overline{v_0 v} \in E(G)} \lambda(v) D_{v_0+v,0}^{(0)} \qquad\qquad D_{v_s,s}^{(s)} \mapsto \sum_{\overline{v_s v} \in E(G)} \lambda(v) D_{v_0+v,0}^{(0)}$$

where $E(G)$ is the set of edges of the graph G.

Proof. This follows directly from the description of $h^{(0)}$ in section 2.2 and proposition 3.16 of [PS08]. □

An important fact is that, in a sense, such a map of Cartier divisors is compatible with blowing up or down. More specifically, let $\pi : X^{\text{tot}} \to S$ be a homogeneous one-parameter deformation of the rational $\mathbb{C}^*$-surface X_0 and let $\phi_0 : X_0 \to X_0'$ be an invariant blowdown of a minus one curve with $E^{(0)}$ the corresponding exceptional divisor. Let π' be the blowdown of π by ϕ, with X_s' the general fiber of π'. From the description of the blowdown of a degeneration diagram, one easily confirms that we have an invariant blowdown $\phi_s : X_s \to X_s'$; let $E^{(s)}$ be the corresponding exceptional divisor.

Proposition 3.11. *In the above situation, $\pi_{s,0}(E^{(s)}) = E^{(0)}$. Furthermore, the following diagram commutes:*

$$\begin{tikzcd}
\text{T-CaDiv}(X_0) & \text{T-CaDiv}(X_s) \arrow[l, "\pi_{s,0}"'] \\
\text{T-CaDiv}(X_0') \arrow[u, "\phi_0^*"] & \text{T-CaDiv}(X_s') \arrow[l, "\pi_{s,0}'"] \arrow[u, "\phi_s^*"']
\end{tikzcd}$$

Proof. The claim regarding the exceptional divisor follows from the description of the blowdown of a degeneration diagram and from proposition 3.10. Indeed, if $(\mathcal{M}, G)$ is the degeneration diagram corresponding to π and $E^{(0)}$ corresponds to some edge e of G, then $E^{(s)}$ corresponds to the vertex of e with degree one, which then obviously maps to the desired divisor, since the other vertex of e must have height one. If on the other hand $E^{(0)}$ is some other divisor of X_0, the claim is immediate from proposition 3.10.

The commutativity of the diagram follows from the description of $h^{(0)}$ in section 2.2. Indeed, the pullback of an invariant Cartier divisor D_h on a T-variety $X(\Xi')$ to some blowup $X(\Xi)$ corresponds to the same piecewise affine function h. Furthermore, one easily sees from the description of $h^{(0)}$ that further refinement in a divisorial fan Ξ does not affect the construction of $h^{(0)}$. $\square$

Example. We look at an explicit description of the map $\bar{\pi}_{s,0}$, where π is a deformation of a Hirzebruch surface. If $X(\mathcal{M}) = \mathcal{F}_r$ and $\mathcal{M}$ admits a non-trivial degeneration diagram, we can assume that $\mathcal{M}_0$ has vertices $-\frac{1}{r+\alpha}$, 0 and that $\mathcal{M}_s$ has vertices $0, \frac{1}{\alpha}$ for some $\alpha > 0$. We call this multidivisor $\mathcal{M}(r,\alpha)$. Note that with the exception of the case $r = 0, \alpha = 1$, there is only one possible graph G making $(\mathcal{M}(r,\alpha), G)$ into a degeneration diagram. Indeed, this is the bipartite graph where both 0 vertices have degree two and the other two vertices have degree one. For the case $r = 0, \alpha = \pm 1$, there is also the possibility of the bipartite graph $\tilde{G}$ where both 0 vertices have degree one and the other two vertices, in this case lattice points, have degree two. In any case, the degeneration diagram $(\mathcal{M}(r,\alpha), G)$ (or $(\mathcal{M}(r,\alpha), \tilde{G})$) has corresponding special fiber $\mathcal{F}_{r+2\alpha}$. The difference between G and $\tilde{G}$ corresponds to a flip on the total space of the deformation.

For any Hirzebruch surface $\mathcal{F}_r$ with $r > 0$, let P be the divisor class of the fiber of the ruling on $\mathcal{F}_r$, and let Q be the unique class with $Q^2 = r$ and $P.Q = 1$. Now considering the isomorphism $X(\mathcal{M}(r,\alpha)) \cong \mathcal{F}_r$, P and Q can respectively be represented by $D^{(s)}_{0,s}$ and $D^{(s)}_{1/\alpha,s}$. Consider now the deformation π from $\mathcal{F}_{r+2\alpha}$ to $\mathcal{F}_r$ determined by the deformation diagram $(\mathcal{M}(r,\alpha), G)$ and assume $r > 0$. Then $\bar{\pi}_{s,0}(P)$ can be represented by $D^{(0)}_{1/\alpha,0}$ and $\bar{\pi}_{s,0}(Q)$ can be represented by $(r+\alpha)D^{(0)}_{-1/(r+\alpha),0} + D^{(0)}_{0,0}$. One easily checks that

$$\bar{\pi}_{s,0}(P) = P \tag{8}$$
$$\bar{\pi}_{s,0}(Q) = Q - \alpha P \tag{9}$$

where by abuse of notation, the P, Q on the right hand side of the equalities represent classes in $\mathrm{Pic}(\mathcal{F}_{r+2\alpha})$.

The case of $r = 0$ requires slightly more care, since there are two possible rulings on $\mathcal{F}_0$. Fix an isomorphism $\mathcal{F}_0 \cong X(\mathcal{M}(0,\alpha))$ and consider the ruling of $\mathcal{F}_0$ given by the quotient map of the $\mathbb{C}^*$-action on $X(\mathcal{M}(0,\alpha))$; note that this doesn't depend on α. Then P and Q can be represented exactly as above. For π corresponding to the degeneration diagram $(\mathcal{M}(0,\alpha), G)$, we once again have equations (8) and (9). On the other hand, for π corresponding to the degeneration diagram $(\mathcal{M}(0,1), \tilde{G})$, we have $\bar{\pi}_{s,0}(P) = Q - P$ and $\bar{\pi}_{s,0}(Q) = P$. Thus, if in this case we instead consider the other possible ruling of $\mathcal{F}_0$ (and thus swap P and Q), we once again have equations (8) and (9).

4 Exceptional Sequences

We shall now turn our attention to exceptional sequences.

4.1 Exceptional Sequences and Toric Systems

In the following all surfaces are smooth and complete. For general features of derived categories in algebraic geometry we refer to [Huy06]. By $\mathcal{D}^b(X)$ we denote the bounded derived category of coherent sheaves on some complex variety X.

Definition. An object E of $\mathcal{D}^b(X)$ is called *exceptional* if it fulfills

$$\operatorname{Ext}^i(E,E) = \begin{cases} \mathbb{C} & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$$

An *exceptional sequence* $\mathcal{E}$ is a finite sequence of exceptional objects $(E_1, \dots, E_n)$ such that there are no morphisms back, that is, $\operatorname{Ext}^k(E_j, E_i) = 0$ for $j > i$ and all k. Such a sequence is called *strongly exceptional* if additionally $\operatorname{Ext}^k(E_j, E_i) = 0$ for all i, j and all $k > 0$. An exceptional sequence is called *full* if $E_1, \dots, E_n$ generate $\mathcal{D}^b(X)$, that is, the smallest full triangulated subcategory of $\mathcal{D}^b(X)$ containing all E_i's is already $\mathcal{D}^b(X)$.

Our focus lies exclusively on full exceptional sequences of line bundles on rational surfaces. So unless explicitly stated otherwise, all exceptional sequences will be full and consist only of line bundles. We identify isomorphism classes of line bundles with classes of divisors, and will thus use additive notation. Furthermore, if $\phi : \tilde{X} \to X$ is some blowup and E is a line bundle on X, we will often use E to denote $\phi^*(E)$ as well, as long as the meaning is clear.

A related concept introduced by Hille and Perling [HP08] are so-called *toric systems*:

Definition. A *toric system* on a rational surface X of Picard number $n-2$ is a sequence of line bundles $\mathcal{A} = (A_1, \dots, A_n)$ such that

(i) $A_i.A_{i+1} = 1$;

(ii) $A_i.A_j = 0$ for $j \notin \{i-1, i, i+1\}$;

(iii) $\sum_{i=1}^n A_i = -K_X$;

where we consider indices cyclically modulo n.

One of the very nice ideas in [HP08] is that from every toric system $\mathcal{A} = (A_1, \dots, A_n)$ on a rational surface X we can construct a toric surface $\operatorname{TV}(\mathcal{A})$. Indeed, by setting $-b_i = \chi(A_i) - 2$, $\operatorname{TV}(\mathcal{A}) := \operatorname{TV}(b_1, \dots b_n)$ is a smooth toric surface. On the other hand, starting with some rational surface X with an exceptional sequence $\mathcal{E}$, we can construct an associated toric system $\mathcal{A}$, by setting

$$A_i := E_{i+1} - E_i \text{ and } A_n := (E_1 - K_X) - E_n. \tag{10}$$

If a toric system $\mathcal{A}$ can be constructed in this manner, we call it *exceptional*. If $\mathcal{A}$ can be constructed in this manner from a strongly exceptional sequence $\mathcal{E}$, we call it *strongly exceptional*. Of course, a toric system $\mathcal{A}$ is (strongly) exceptional if and only if $(0, A_1, A_1 + A_2, \dots, \sum A_i)$

forms a (strongly) exceptional sequence; in such cases we call this the exceptional sequence associated to $\mathcal{A}$. Similarly, $\mathcal{A}$ is (strongly) exceptional if and only if

$$H^l(X, \sum_{i=j}^{k} -A_i) = 0 \qquad \big(\text{and } H^{l+1}(X, \sum_{i=j}^{k} A_i) = 0\big)$$

for all $l \geq 0$ and $1 \leq j \leq k < n$. Any cyclic permutation or reflection of the indices takes an (exceptional) toric system to an (exceptional) toric system and doesn't change the associated toric variety. Finally, note that if X is a toric surface, the invariant divisors D_i properly ordered form a canonical exceptional toric system $\mathcal{A} = (D_1, \ldots, D_n)$. In this case, we have $\mathrm{TV}(\mathcal{A}) = X$.

4.2 Augmented Toric Systems

It follows from proposition 3.3 that any homogeneous deformation of rational $\mathbb{C}^*$-surfaces can be attained by repeatedly blowing up a deformation of a Hirzebruch surface. For exceptional sequences, the situation is somewhat similar: In [HP08], a construction called *augmentation* was established, which constructs new toric systems from blowups. We recall this notion, adapting notation slightly.

Definition. Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a toric system on a rational surface X and $\tilde{X} \to X$ a blowup in one point with exceptional divisor R. Then for every $1 \leq i \leq n$ we can construct a toric system on $\tilde{X}$

$$\mathrm{Aug}_i\, \mathcal{A} = (A_1, \ldots, A_{i-1}, A_i - R, R, A_{i+1} - R, A_{i+2}, \ldots, A_n)$$

called the *augmentation* of $\mathcal{A}$ at the position i. For any sequence of blowups, let $\mathrm{Aug}_{i_n,\ldots,i_1}\, \mathcal{A}$ denote the repeated augmentation of a toric system $\mathcal{A}$ at the positions $i_1, \ldots, i_n$.[4]

Lemma 4.1 ([HP08])**.** *The augmentation of any exceptional toric system is an exceptional toric system.*

Lemma 4.2. *Let $\mathcal{A}$ be a toric system with* $\mathrm{TV}(\mathcal{A}) = \mathrm{TV}(b_1, \ldots, b_n)$. *Then*

$$\mathrm{TV}(\mathrm{Aug}_i\, \mathcal{A}) = \mathrm{TV}(b_1, \ldots, b_{i-1}, b_i + 1, 1, b_{i+1} + 1, b_{i+2}, \ldots, b_n).$$

In other words, augmenting at position i results in a blowup of the associated toric variety $\mathrm{TV}(\mathcal{A})$ *at the same position (i.e. inserting a new ray between the rays labeled with i and $i+1$).*

Proof. This statement can be shown by straightforward computation, using the formula $\chi(D) = 1 + \frac{1}{2}(D^2 - K.D)$ for the Euler characteristic. □

On the Hirzebruch surfaces $\mathcal{F}_r$ we will always choose the basis of $\mathrm{Pic}(\mathcal{F}_r)$ used in the final example of section 3.4, that is, classes $P, Q \in \mathrm{Pic}(\mathcal{F}_r)$ with P the class of the fiber of ruling on $\mathcal{F}_r$ and Q such that $Q^2 = r$ and $P \cdot Q = 1$ (so P and Q are the generators of the nef cone). Note that for $\mathcal{F}_0$, P and Q are interchangeable. Hille and Perling have calculated all possible toric systems on Hirzebruch surfaces:

[4] Here we are only looking at what Hille and Perling call standard augmentations

Proposition 4.3 ([HP08, Proposition 5.2]). *All toric systems on $\mathcal{F}_r$ up to cyclic permutation or reflection of the indices are of the form*

$$\mathcal{A}_{r,i} = \big(P, iP+Q, P, -(r+i)P+Q\big);\ \textit{and}$$
$$\tilde{\mathcal{A}}_{r,i} = \big(-\frac{r}{2}P+Q, P+i(-\frac{r}{2}P+Q), -\frac{r}{2}P+Q, P-i(-\frac{r}{2}P+Q)\big)\ \textit{if } r \textit{ is even.}$$

$\mathcal{A}_{r,i}$ is always exceptional, and strongly exceptional if and only if $i \geq 1$. $\tilde{\mathcal{A}}_{r,i}$ is exceptional only if $r=0$, or if $r=2$ and $i=0$. Finally,

$$\mathrm{TV}(\mathcal{A}_{r,i}) = \mathcal{F}_{|r+2i|} \qquad \textit{and} \qquad \mathrm{TV}(\tilde{\mathcal{A}}_{r,i}) = \mathcal{F}_{|2i|}.$$

Now let X be any rational surface of Picard number $\rho \geq 2$. We call an exceptional toric system $\mathcal{A}_X$ on X *tame* if there is some sequence of blowups $X = X^n \to \cdots \to X^0 = \mathcal{F}_r$ such that $\mathcal{A}_X$ can be constructed inductively by augmenting some exceptional toric system on $\mathcal{F}_r$. Hille and Perling conjectured that all exceptional toric systems are in fact tame toric systems. One small piece of evidence for this is the following proposition:

Proposition 4.4. *Let Y be a toric surface with the same Picard rank $\rho > 2$ as a rational surface X. Then there is an tame exceptional toric system $\mathcal{A}$ on X with* $\mathrm{TV}(\mathcal{A}) = Y$.

Proof. For X and Y there is a sequence of blowups reducing to Hirzebruch surfaces $\mathcal{F}_r$ and $\mathcal{F}_s$ respectively, say $X = X^n \to \cdots \to X^0 = \mathcal{F}_r$ and $Y = Y^n \to \cdots \to Y^0 = \mathcal{F}_s$.

Assume that $r = s \mod 2$. Let $\mathcal{A}$ be the toric system on X attained by repeatedly augmenting $\mathcal{A}_{r,(s-r)/2}$ at the same positions where we blow up $\mathcal{F}_s$ to get to Y. Due to lemma 4.2 it follows that $\mathrm{TV}(\mathcal{A}) = Y$.

Suppose instead that $r \neq s \mod 2$. Note that the blowdown $Y^1 \to \mathcal{F}_s$ isn't unique; there is also a blowdown $Y_1 \to \mathcal{F}_{s'}$ for either $s' = s+1$ or $s' = s-1$. Thus, by taking instead the blowdown $Y_1 \to \mathcal{F}_{s'}$ we can in fact assume that $r = s \mod 2$. ☐

4.3 Exceptional Sequences and Deformations

We now consider the behaviour of exceptional sequences under homogeneous deformations. In what follows, all surfaces will be rational and have a $\mathbb{C}^*$-action, and we will only consider homogeneous deformations. Consider thus any homogeneous deformation π of rational $\mathbb{C}^*$-surfaces from X_0 to X_s. Let $\mathcal{A} = (A_1, \ldots, A_n)$ be any n-tuple of line bundles on X_s. Then we define $\bar{\pi}_{s,0}(\mathcal{A})$ to be the n-tuple $(\bar{\pi}_{s,0}(A_1), \ldots, \bar{\pi}_{s,0}(A_n))$.

Our first observation is that degeneration preserves toric systems:

Theorem 4.5. *Let π be a deformation of rational $\mathbb{C}^*$-surfaces from X_0 to X_s and let $\mathcal{A}$ be a toric system on X_s. Then $\bar{\pi}_{0,s}(\mathcal{A})$ is a toric system, and* $\mathrm{TV}(\mathcal{A}) = \mathrm{TV}(\bar{\pi}_{s,0}(\mathcal{A}))$. *Moreover, let π' be a blowup of this deformation as in proposition 3.3. Then*

$$\mathrm{Aug}_i\, \bar{\pi}_{s,0}(\mathcal{A}) = \bar{\pi}'_{s,0}(\mathrm{Aug}_i\, \mathcal{A}). \tag{11}$$

In other words, augmentation commutes with degeneration.

Proof. The fact that $\bar{\pi}_{s,0}(\mathcal{A})$ is a toric system is immediate, since $\bar{\pi}_{s,0}$ preserves intersection numbers and the canonical class, see respectively theorems 2.6 and 2.7. Furthermore, the equality $\mathrm{TV}(\mathcal{A}) = \mathrm{TV}(\bar{\pi}_{s,0}(\mathcal{A}))$ automatically follows from theorem 2.5. Finally, equation (11) follows directly from proposition 3.11. $\square$

On the other hand, we can make a much stronger statement concerning the behavior or toric systems under deformation:

Theorem 4.6. *Let π be a deformation of rational $\mathbb{C}^*$-surfaces from X_0 to X_s and let $\mathcal{A}$ be a toric system on X_0. Then $\bar{\pi}_{s,0}^{-1}(\mathcal{A})$ is a toric system, and $\mathrm{TV}(\mathcal{A}) = \mathrm{TV}(\bar{\pi}_{s,0}^{-1}(\mathcal{A}))$. Furthermore, if $\mathcal{A}$ is exceptional/tame/strongly exceptional, then so is $\bar{\pi}_{s,0}^{-1}(\mathcal{A})$.*

Proof. The first two statements are shown exactly as in the proof of theorem 4.5. From theorem 2.5 we have that $\bar{\pi}_{s,0}^{-1}$ preserves the vanishing of cohomology, which implies that $\bar{\pi}_{s,0}^{-1}$ preserves (strong) exceptionality. Finally, if $\mathcal{A}$ is tame, we can blow down X_0 to some X_0' such that $\mathcal{A} = \mathrm{Aug}_i\,\mathcal{A}'$ for some tame toric system $\mathcal{A}'$ on X_0'. We then blown down π to π' as in proposition 3.3, which induces a blowdown $X_s \to X_s'$. If $(\bar{\pi}'_{s,0})^{-1}(\mathcal{A}')$ is tame, then $\bar{\pi}_{s,0}^{-1}(\mathcal{A})$ is as well, since by theorem 4.5 we have that

$$\bar{\pi}_{s,0}^{-1}(\mathcal{A}) = \mathrm{Aug}_i(\bar{\pi}'_{s,0})^{-1}(\mathcal{A}')$$

with respect to the blowup $X_s \to X_s'$. The statement then follows by induction on the Picard number of X_0. $\square$

Combining the above two theorems with theorem 3.5 provides us with a proof of our main theorem 2 from the introduction. We now turn our attention to Hirzebruch surfaces, where we have some more explicit results. The first is the following proposition:

Proposition 4.7. *Consider some homogeneous deformation π from $\mathcal{F}_{r+2\alpha}$ to $\mathcal{F}_r$ for $r > 0$. Then for all i,*

$$\bar{\pi}_{s,0}(\mathcal{A}_{r,i}) = \mathcal{A}_{r+2\alpha,i-\alpha}$$

and if r is even,

$$\bar{\pi}_{s,0}(\tilde{\mathcal{A}}_{r,i}) = \tilde{\mathcal{A}}_{r+2\alpha,i}.$$

In particular, exceptional toric systems degenerate to exceptional toric systems.

Proof. We recorded all possible deformations at the end of section 3.4. As noted there, we have:

$$\begin{array}{rcl} \bar{\pi}_{s,0} : \mathrm{Pic}(\mathcal{F}_r) & \to & \mathrm{Pic}(\mathcal{F}_{r+2\alpha}) \\ P & \mapsto & P \\ Q & \mapsto & Q - \alpha P \end{array}$$

The proposition then follows from direct calculation. $\square$

Remark. It might seem odd that in the above theorem, we must rule out the case $X_s = \mathcal{F}_0$. This is due to the interchangeable roles of P and Q in the basis of $\mathrm{Pic}(\mathcal{F}_0)$. In this case, either $\bar{\pi}_{s,0}(\mathcal{A}_{0,i}) = \mathcal{A}_{2\alpha,i-\alpha}$ as above or $\bar{\pi}_{s,0}(\mathcal{A}_{0,i}) = \tilde{\mathcal{A}}_{2\alpha,i-\alpha}$, and either $\bar{\pi}_{s,0}(\tilde{\mathcal{A}}_{0,i}) = \tilde{\mathcal{A}}_{2\alpha,i}$ as above or $\bar{\pi}_{s,0}(\tilde{\mathcal{A}}_{0,i}) = \mathcal{A}_{2\alpha,i-\alpha}$.

We can further explain the above situation on Hirzebruch surfaces in terms of so-called *mutations*. We first recall their definition from [Rud90]:

Definition. Let (E, F) be a (not necessarily full) exceptional sequence of two arbitrary objects in $\mathcal{D}^b(X)$. The *left mutation* $L_F E$ of F by E is an object of $\mathcal{D}^b(X)$ that fits into the triangle

$$L_E F \to \bigoplus_l \operatorname{Hom}(E, F[l]) \otimes E[-l] \stackrel{can}{\to} F \to L_E F[1].$$

Similarly, we define the *right mutation* $R_F E$ of F by E as the object that fits into the triangle

$$E \stackrel{can^*}{\to} \bigoplus_l \operatorname{Hom}(E, F[l])^*[l] \otimes F \to R_F E \to E[1].$$

For an exceptional sequence $\mathcal{E} = (E_1, \dots, E_n)$ of arbitrary objects we define the left mutation of $\mathcal{E}$ at position i as

$$L_i \mathcal{E} = (E_1, \dots, E_{i-1}, L_{E_i} E_{i+1}, E_i, E_{i+2}, \dots, E_n),$$

and analogously the right mutation of $\mathcal{E}$ at position i is

$$R_i \mathcal{E} = (E_1, \dots, E_{i-1}, E_{i+1}, R_{E_{i+1}} E_i, E_{i+2}, \dots, E_n).$$

Remark. In [Rud90] it is shown that the left and right mutations of an exceptional sequence are again exceptional, and that the right and left mutation are inverses of each other.

On Hirzebruch surfaces, it is possible to mutate an exceptional sequence of line bundles such that the mutation still consists of line bundles. According to the proposition 4.3 the exceptional sequences on $\mathcal{F}_r$, $r > 0$ correspond to toric systems of the form $\mathcal{A}_{r,i}$ up to cyclic permutation or reflection of indices. Now let $\mathcal{E}$ be an exceptional sequence with toric system $\mathcal{A}_{r,i}$. Consider a mutation of $\mathcal{E}$ at the first position. To calculate this, we must look at

$$\bigoplus_l \operatorname{Hom}(\mathcal{O}, \mathcal{O}(P)[l]) \otimes \mathcal{O}[-l] \stackrel{can}{\to} \mathcal{O}(P).$$

Since

$$\hom(\mathcal{O}, \mathcal{O}(P)[l]) = h^l(X, \mathcal{O}(P)) = \begin{cases} 2 & l = 0 \\ 0 & l \neq 0 \end{cases}$$

and the map *can* is surjective, $L_{\mathcal{O}} \mathcal{O}(P)$ is just the ordinary kernel of this map. In fact, $L_{\mathcal{O}} \mathcal{O}(P) = \mathcal{O}(-P)$, so

$$L_1 \mathcal{E} = (-P, 0, (i+1)P + Q, (i+2)P + Q).$$

Thus, on the level of toric systems, the left mutation of the toric system at the first position is

$$L_1 \mathcal{A}_{r,i} = \mathcal{A}_{r,i+1}.$$

Since the first element of the mutated toric system is again P, we can iterate this process. Hence, we denote by $L_1^\alpha \mathcal{A}$ the result of left mutating $\mathcal{A}$ α-times. Note that we can extent

this notion also to $\alpha \in \mathbb{Z}$. Combining this with the previous proposition gives us for any deformation π from $\mathcal{F}_{r+2\alpha}$ to $\mathcal{F}_r$

$$\bar{\pi}_{s,0}(L_1^{\alpha}\mathcal{A}_{r,i}) = \mathcal{A}_{r+2\alpha,i}.$$

In particular, the α-fold left mutation of the canonical toric system on $\mathcal{F}_r$ degenerates to the canonical toric system on $\mathcal{F}_{r+2\alpha}$. Likewise, changing to the viewpoint of deformation, we have

$$\bar{\pi}_{s,0}^{-1}(L_1^{-\alpha}\mathcal{A}_{r+2\alpha,i}) = \mathcal{A}_{r,i}.$$

Although we originally ruled out the case that $r = 0$, note that this isn't really necessary. We just need to choose the basis of $P, Q \in \mathrm{Pic}(\mathcal{F}_0)$ such that $\bar{\pi}_{s,0}(P) = P$.

We can extend the above discussion on Hirzebruch surfaces to general rational $\mathbb{C}^*$-surfaces as follows:

Theorem 4.8. *Consider a homogeneous deformation or degeneration of rational $\mathbb{C}^*$-surfaces from X to a toric surface Y, which blows down to a deformation respectively degeneration from $\mathcal{F}_r$ to $\mathcal{F}_{r+2\alpha}$ for $\alpha \in \mathbb{Z}$. Consider the augmentation with respect to this blowdown $\mathcal{A}_Y = \mathrm{Aug}_{i_n,\ldots,i_1} \mathcal{A}_{r+2\alpha,0}$ such that $\mathcal{A}_Y$ is the canonical toric system on Y.*[5] *Then the toric system*

$$\mathcal{A} = \mathrm{Aug}_{i_n,\ldots,i_1} L_1^{\alpha}\mathcal{A}_{r,0}$$

on X deforms respectively degenerates to $\mathcal{A}_Y$.

Proof. Combine the case for Hirzebruch surfaces discussed above with equation (11) from theorem 4.5. ☐

Although we would like to claim that homogeneous degenerations preserve exceptional and strongly exceptional toric systems, this is simply not the case. First, consider the exceptional toric system $\mathcal{A}_{0,i}$ on $\mathcal{F}_0$. As noted above, this can be degenerated to $\tilde{\mathcal{A}}_{2\alpha,i-\alpha}$, which is not exceptional if $\alpha > 1$. Thus, exceptional toric systems are not preserved under degeneration. Similarly, consider the strongly exceptional toric system $\mathcal{A}_{r,i}$ on $\mathcal{F}_r$, where $i \geq 1$. As we saw above, this can be degenerated to $\mathcal{A}_{r+2\alpha,i-\alpha}$ on $\mathcal{F}_{r+2\alpha}$, which is no longer strongly exceptional if $i < \alpha + 1$.

However, the situation isn't hopeless—for any degeneration, we can identify a subset of exceptional toric systems which degenerate to exceptional toric systems:

Definition. Let π be a homogeneous deformation of C^*-surfaces $X_0 \rightsquigarrow X_s$, and let $\mathcal{A}$ be a tame toric system on X_s. We say that $\mathcal{A}$ is *compatible* with π if:

(i) X_s is a Hirzebruch surface and $\bar{\pi}_{s,0}(\mathcal{A})$ is exceptional; or

(ii) There is a blowdown of π to π' inducing a blowdown $X_s \to X'_s$ such that $\mathcal{A}$ is an augmentation of a toric system $\mathcal{A}'$ on X'_s compatible with π'.

[5] If either $r = 0$ or $r + 2\alpha = 0$, we must choose the basis P, Q of $\mathrm{Pic}(\mathcal{F}_0)$ as above.

Proposition 4.7 and the following remark thus give us an explicit description of the toric systems compatible with any deformation of Hirzebruch surfaces. The second condition above then can be applied inductively to determine all toric systems compatible with a given deformation. The importance of compatibility is made clear by the following theorem and corollary:

Theorem 4.9. *Let π be a homogeneous deformation of rational $\mathbb{C}^*$-surfaces $X_0 \rightsquigarrow X_s$ and $\mathcal{A}$ a tame toric system on X_s. Then $\mathcal{A}$ is compatible with π if and only if $\bar{\pi}_{s,0}(\mathcal{A})$ is a tame toric system.*

Proof. We first prove that if $\mathcal{A}$ is compatible with π, then $\bar{\pi}_{s,0}(\mathcal{A})$ is tame; this is done by induction on the Picard number ρ of X_0. The case $\rho = 2$ follows directly from the definition of compatibility. On the other hand, the induction step follows from equation (11) and lemma 4.1.

Now suppose that $\mathcal{A}$ isn't compatible with π, but $\mathcal{A}_{X_0} := \bar{\pi}_{s,0}(\mathcal{A})$ is tame. Then there is a blowdown $X_0 \to X_0'$ and a tame toric system $\mathcal{A}_{X_0'}$ on X_0' such that $\mathcal{A}_{X_0}$ is an augmentation of $\mathcal{A}_{X_0'}$ with respect to this blowup. The blowdown $X_0 \to X_0'$ induces a unique blowdown of π to some π' with special fiber X_0' and some general fiber X_s'. If we set $\mathcal{A}' := (\bar{\pi}_{s,0}')^{-1}(\mathcal{A}_{X_0'})$, then by equation (11) we can conclude that $\mathcal{A}$ is an augmentation of $\mathcal{A}'$, and that if $\mathcal{A}'$ is compatible with π', then $\mathcal{A}$ must be compatible with π. Applying induction, we arrive at a contradiction, and can thus conclude that $\mathcal{A}_{X_0}$ must not have been tame. ∎

Corollary 4.10. *Assume that all exceptional toric systems are tame. In the above setting, $\mathcal{A}$ is compatible with π if and only if $\bar{\pi}_{s,0}(\mathcal{A})$ is exceptional.*

Proof. This is immediate from the above theorem. ∎

Remark. It is not difficult to find tame toric systems which are not compatible with certain deformations. Indeed, consider any toric surface X_s with multiple invariant minus one curves, and let $\mathcal{A}$ be a tame toric system on X_s such that $\mathrm{TV}(\mathcal{A})$ only has a single invariant minus one curve. Then we claim there is a degeneration of X_s with which $\mathcal{A}$ is not compatible. Indeed, since $\mathrm{TV}(\mathcal{A})$ only has a single invariant minus one curve, there exists a unique blowdown $X_s \to X_s'$ such that $\mathcal{A}$ is the augmentation of a tame toric system on X_s'; let C be the corresponding minus one curve. Now let π be any degeneration of X_s to some X_0 such that the vertex v corresponding to C has at least degree 2 in the corresponding degeneration diagram. Then $\mathcal{A}$ is not compatible with π, since π cannot be blown down to have general fiber X_s'.

4.4 Noncommutative Deformations

If $\mathcal{E} = (E_1, \ldots, E_n)$ is a full strongly exceptional sequence (not necessarily of line bundles) on a variety X, then the corresponding *tilting sheaf* is defined to be

$$\mathcal{T} = \bigoplus E_i.$$

There is then an equivalence of categories between $\mathcal{D}^b(X)$ and $\mathcal{D}^b(\mathrm{End}(\mathcal{T})\text{-mod})$, see [Bon89]. The algebra $\mathrm{End}(\mathcal{T})$ can be described as a finite path algebra with relations, see [Per09] for

examples. Note that if $\mathcal{E}$ is a full strongly exceptional sequence of line bundles on a rational surface X, then we have

$$\mathrm{End}(\mathcal{T}) = \bigoplus_{j \le k < n} H^0\Big(X, \sum_{i=j}^{k} A_i\Big)$$

where $\mathcal{A}$ is the toric system corresponding to $\mathcal{E}$.

Let Γ be a family of algebras parametrized over a base variety S such that for every $s \in S$, the corresponding algebra Γ_s has the form $\mathrm{End}(\mathcal{T})$ for some tilting sheaf $\mathcal{T}$ on some variety. We loosely call the family Γ a *noncommutative deformation* as it offers a way of "deforming" varieties via derived categories. Several concrete examples are presented in section 7 of [Per09]. Now suppose that $\pi : X^{\mathrm{tot}} \to S$ is a homogeneous deformation of a rational $\mathbb{C}^*$-surface X_0 and $\mathcal{E}$ is a full strongly exceptional sequence of lines bundles on X_0. This data naturally gives rise to a noncommutative deformation. Indeed, for $s \in S$, set

$$\Gamma_s = \mathrm{End}\left(\bigoplus \bar{\pi}_{s,0}^{-1}(E_i)\right) = \bigoplus_{j \le k < n} H^0\Big(X_s, \sum_{i=j}^{k} \bar{\pi}_{s,0}^{-1}(A_i)\Big)$$

where $\mathcal{A}$ is the corresponding toric system and by $\bar{\pi}_{0,0}$ we simply mean the identity.

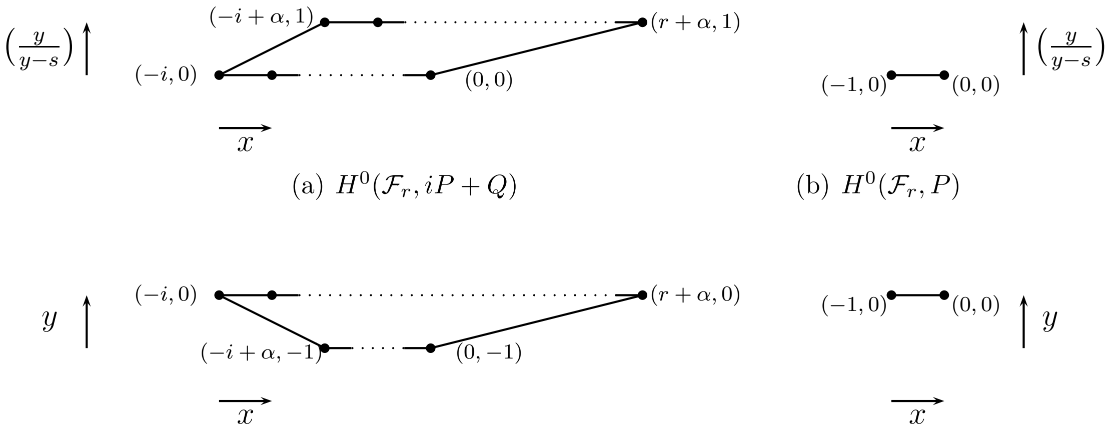
We wish to describe such families explicitly in the case of Hirzebruch surfaces by defining a family of quivers. Fix some $r \ge 0$ and $i > 0$, and $0 < \alpha < i$. We then have the deformation π from $\mathcal{F}_{r+2\alpha}$ to $\mathcal{F}_r$ given by the degeneration diagram $(\mathcal{M}(r,\alpha), G)$ from the example at the end of section 3.4. Furthermore, the toric system $\mathcal{A} = \mathcal{A}_{r+2\alpha, i-\alpha}$ corresponds to a strongly exceptional sequence on $\mathcal{F}_{r+2\alpha}$, and for $s \in S^*$, $\bar{\pi}_{s,0}^{-1}(\mathcal{A}) = \mathcal{A}_{r,i}$. To calculate Γ_s for $s \in S$, we thus need to know the cohomology groups $H^0(\mathcal{F}_r, iP + Q)$, $H^0(\mathcal{F}_r, P)$, $H^0(\mathcal{F}_{r+2\alpha}, (i-\alpha)P + Q)$, and $H^0(\mathcal{F}_{r+2\alpha}, P)$. These can be calculated using standard toric methods. We will represent P and Q as divisors on $\mathcal{F}_r$ and $\mathcal{F}_{r+2\alpha}$ as we have in section 3.4. That is, on $\mathcal{F}_r$ we represent P and Q by respectively $D_{0,s}^{(s)}$ and $D_{1/\alpha,s}^{(s)}$, and on $\mathcal{F}_{r+2\alpha}$ we represent P and Q by respectively $D_{1/\alpha,s}^{(0)}$ and $(r+\alpha)D_{-1/(r+\alpha),s}^{(0)} + D_{0,0}^{(0)} + \alpha D_{1/\alpha,0}^{(0)}$. In figure 7, we present polytopes where each lattice point corresponds to a monomial element of the basis of the relevant cohomology group. Note that for $s \neq 0$, instead of having monomials in the usual variables x and y we have monomials in the variables x and $\frac{y}{y-s}$.

We first concern ourselves with the global sections of $iP + Q$ and $(i-\alpha)P + Q$ on $\mathcal{F}_r$ and $\mathcal{F}_{r+2\alpha}$, respectively. For $s \in S$, let $b_j^{(s)} = x^j$ and let $d_j^{(s)} = x^j \frac{y}{y-s}$. Note that for $s \neq 0$, we have that $b_{-i}^{(s)}, \ldots, b_0^{(s)}, d_{-i+\alpha}^{(s)}, \ldots, d_{r+\alpha}^{(s)}$ is a basis for $H^0(\mathcal{F}_r, iP + Q)$. However, for $s = 0$ we run into difficulties, since $b_j^{(0)} = d_j^{(0)}$. Now set

$$c_j^{(s)} = \frac{1}{s}(d_j^{(s)} - b_j^{(s)}).$$

Note then that $c_j^{(0)} = y^{-1}$. It follows that

$$b_{-i}^{(s)}, \ldots, b_0^{(s)}, c_{-i+\alpha}^{(s)}, \ldots, c_0^{(s)}, d_1^{(s)}, \ldots, d_{r+\alpha}^{(s)}$$



(a) $H^0(\mathcal{F}_r, iP+Q)$ (b) $H^0(\mathcal{F}_r, P)$

(c) $H^0(\mathcal{F}_{r+2\alpha}, (i-\alpha)P+Q)$ (d) $H^0(\mathcal{F}_{r+2\alpha}, P)$

Figure 7: Global Sections of Toric Systems on $\mathcal{F}_r$ and $\mathcal{F}_{r+2\alpha}$

is a basis of $H^0(\mathcal{F}_r, iP+Q)$ for $s \neq 0$ and of $H^0(\mathcal{F}_{r+2\alpha}, (i-\alpha)P+Q)$ for $s = 0$. Furthermore, this basis is compatible with π insofar as

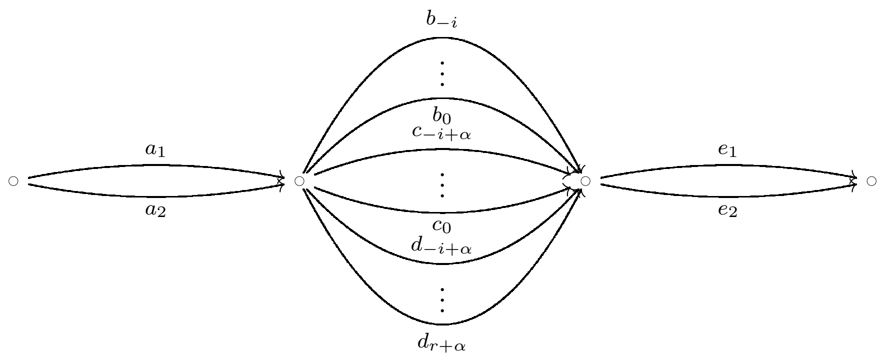
$$\pi_{s,s'}(\operatorname{div}(b_j^{(s)})) = \operatorname{div}(b_j^{(s')})$$
$$\pi_{s,s'}(\operatorname{div}(c_j^{(s)})) = \operatorname{div}(c_j^{(s')})$$
$$\pi_{s,s'}(\operatorname{div}(d_j^{(s)})) = \operatorname{div}(d_j^{(s')})$$

for $s \in S^*$ and $s' \in S$. On the other hand, the monomials x^{-1} and x form a basis of both $H^0(\mathcal{F}_r, P)$ and $H^0(\mathcal{F}_{r+2\alpha}, P)$, which is compatible with π in the same sense.

Figure 8 illustrates a family of quivers. We claim that the corresponding family of path algebras is in fact the desired noncommutative deformation. Indeed, fix some $s \in S$. The paths b_j, c_j, and d_j correspond to the global sections $b_j^{(s)}$, $c_j^{(s)}$, $e_j^{(s)}$ of $iP+Q$ or $(i-\alpha)P+Q$, whereas a_1, a_2, d_1, d_2 correspond to the global sections x^{-1}, x, x^{-1}, x of P. One easily checks that for fixed s, the relations in figure 8 are precisely the relations needed to represent Γ_s. Note that in particular, for $s \neq 0$ we can write c_j in terms of b_j and d_j, and for $s = 0$ we have the relation $b_j = d_j$.

In section 7 of [Per09], there is a similar parameterization of path algebras corresponding to Hirzebruch surfaces, which has the advantage that it can contain arbitrarily many Hirzebruch surfaces. However, our construction has the nice attribute that it directly corresponds to a real deformation. The construction we presented here can in fact be easily generalized to construct a noncommutative deformation coming from an arbitrary homogeneous deformation of rational $\mathbb{C}^*$-surfaces and a strongly exception sequence $\mathcal{E}$ on the special fiber.

Let π be an arbitrary homogeneous deformation of some rational $\mathbb{C}^*$-surface X_0, and consider some exceptional sequence $\mathcal{E}$ on X_0, which isn't strongly exceptional. In this case, $\mathcal{E}$ no longer defines a tilting sheaf. However, exactly as in the strongly exceptional case, we



$$
\begin{aligned}
b_j a_1 &= b_{j-1} a_2 & e_1 b_j a_2 &= e_2 b_j a_1 & & \\
c_j a_1 &= c_{j-1} a_2 & e_1 c_j a_2 &= e_2 c_j a_1 & e_1 b_{j+1} a_2 &= e_2 b_j a_2 \\
d_j a_1 &= d_{j-1} a_2 & e_1 d_j a_2 &= e_2 d_j a_1 & e_1 c_{j+1} a_2 &= e_2 c_j a_2 \\
e_1 b_j &= e_2 b_{j-1} & e_1 b_{j-1} a_2 &= e_1 b_j a_1 & e_1 d_{j+1} a_2 &= e_2 d_j a_2 \\
e_1 c_j &= e_2 c_{j-1} & e_1 c_{j-1} a_2 &= e_1 c_j a_1 & s \cdot c_j &= d_j - b_j \\
e_1 d_j &= e_2 d_{j-1} & e_1 d_{j-1} a_2 &= e_1 d_j a_1 & &
\end{aligned}
$$

Figure 8: A family of quivers

can still define a family of path algebras Γ by

$$\Gamma_s = \bigoplus_{j \le k < n} H^0\Big(X_s, \sum_{i=j}^{k} \bar{\pi}_{s,0}^{-1}(A_i)\Big)$$

where as before, $\mathcal{A}$ is the toric system corresponding to $\mathcal{E}$. Now it is however in general not possible to reconstruct a fiber X_s of the deformation from the corresponding algebra Γ_s. For example, take the same deformation of $\mathcal{F}_{r+2\alpha}$ to $\mathcal{F}_r$ that we used above, but consider now the toric system $\mathcal{A} = \mathcal{A}_{r+2\alpha, i-\alpha}$ for $i < -2$. Then one easily checks that the corresponding family of algebras Γ can be represented by the constant family of path algebras pictured in figure 9. This constant family no longer differentiates between the fibers $\mathcal{F}_{r+2\alpha}$ and $\mathcal{F}_r$.

Figure 9: A trivial family of quivers

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